Waves and currents interacting in a 3d field

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Abstract

The present paper reports on a study of the interaction of a current-free monochromatic surface wave field with a wave-free uniform current field in a three-dimensional flow frame. The formulation of the wave-current field is done under the assumption of irrotational and inviscid flow. We have developed the three dimensional expressions describing the characteristics of the combined flow in terms of mass, momentum and energy transport. These equations are found efficient to describe the sought-for combined wave-current field. The parameters describing the wave-current field after the interaction are the surface disturbance amplitude and length, mean water depth, mean current-like parameter and direction of the combined flow, which would be calculated from a set of equations that satisfy conservation of mean mass, momentum and energy flux and a dispersion relation on the free surface before and after the interaction. The results are shown in terms of relative changes in surface disturbance heights and lengths, current-like parameters and final directions obtained for the combined wave-current field with respect to current-free wave and wave-free current pre-interaction parameters.

Introduction

Interaction of waves and currents is an important topic among researchers and scientists interested in ocean related problems. This is because both the wave and current and consequently their interaction play predominant roles in most of the nearshore and offshore dynamic processes. This includes, for example, stability of structures present in such flow fields, sediment transport and its resulting beach topographic change in the nearshore, characteristics of the navigation channel and reliability of the natural or artificial structures in the offshore zone.

By considering the continuity of momentum flux in a normally incident wave train Longuet-Higgins and Stewart (1960 and 1961), Whitham (1962) derived theoretical expressions for the changes in sea level and other linear and nonlinear characteristics of 2D wave trains. Kemp and Simons (1982 and 1983) described the wave-current interactions for following and reverse current in their successive two papers. Hedges and Lee (1991) showed that an equivalent uniform current under the conditions of approximate constant current vorticity could replace a depth varying current. No attempt is here made for an exhaustive review of the wave-current interaction literature.

In Baddour and Song (1990a and 1990b) a vertically 2D combined wave-current field is postulated to exist as the result of the interaction of collinear, apriori known plane current-free wave and wave-free current fields. When a wave encounters a uniform current, they interact hence generating what is referred to as a wave-current field. Neglecting dissipation, it is assumed that after the interaction, a stable, uniform and irrotational combined wave-current field evolves. This field is expressed in terms of a wave-like surface-disturbance and a current-like component. Conservation of mass, momentum and energy flux before and after the interaction are used to estimate the parameters of the resulting combined wave-current field in 2D and collinear with the pre-interacting fields. In the present paper we present the results of extending Baddour and Song (1990)'s work. The condition of collinearity of the current-free wave, wave-free current and combined wave-current fields is relaxed to formulate in 3D the basic equations that describe the direction and characteristic parameters of the 3D wave-current field. These parameters are computed on satisfying conservation of mean mass, momentum and energy flux and a dispersion relation on the free surface of the flow in 3D.

Formulation of wave-current flow potential in 3D

We assume that a current-free monochromatic plane surface wave of wavelength L_o ($=2\pi / k_o$), height H_o ($=2a_o$) and period T propagates over a water body of depth d_o in the direction given by N_w and that independently there exists a horizontal uniform wave-free current U_o over the same water depth d_o in the direction N_C. When these two plane fields meet, see Fig 1, a plane wave-current combined field develops in the direction ***N***, with a new set of unknown parameters namely, wavelength L $(=2\pi / k)$, height H $(=2a)$, current parameter U and depth d. These unknown parameters together with direction $\vec{N}$ are required to be computed from a system of conservation equations described in the next section. We first formulate the potential of a wave-current field in a direction $\vec{N}$.

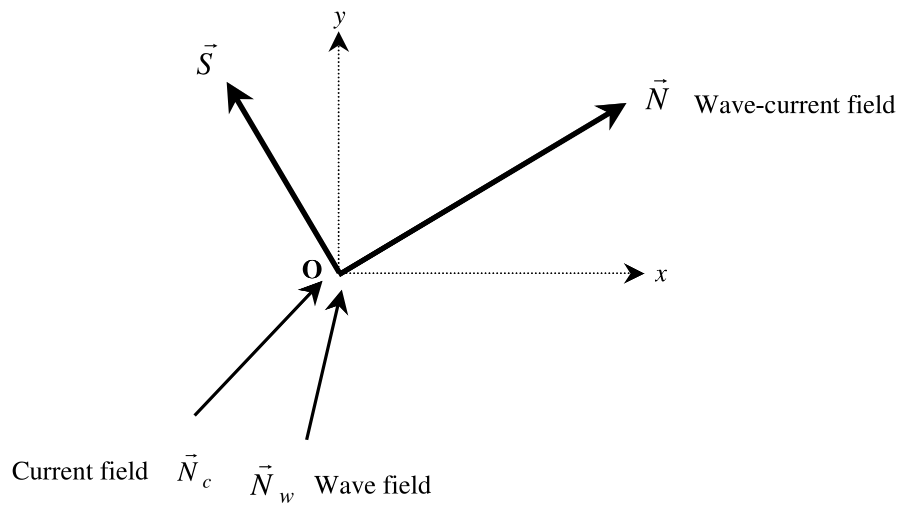

Fig.1 Wave-free current, current-free wave and wave-current fields relative directions

Fig. 1 shows the plan view of the computational domain with O the origin of the 3D inertial frame. The x and y axes subtend the horizontal plane, and z the vertical axis is perpendicular at O to both x and y, and points towards the reader. The unit vectors $\vec{N}_c$, $\vec{N}_w$ and $\vec{N}$ denote the directions of the wave-free current, current-free wave and wave-current plane fields, respectively. The unit vector $\vec{S}$ is normal to $\vec{N}$.

Assuming inviscid and incompressible fluid flows we posit that the result of the interaction between a current-free wave with a wave-free current exists and is here called a wave-current flow field in the $\vec{N}$ direction. This field is described by a velocity potential, given by the following expression to second order in the surface undulation amplitude:

$$\Phi(x,y,z,t) = \vec{U}\cdot\vec{x} + \frac{a_1}{k\sinh kd}\left(\sigma - \vec{U}\cdot\vec{k}\right)\cosh k(d+z)\sin(\vec{k}\cdot\vec{x}-\sigma t) + \frac{1}{k\sinh 2kd}\left(a_2 - \frac{1}{4}a_1^2 k\coth kd\right)\left(\sigma - \vec{U}\cdot\vec{k}\right)\cosh 2k(d+z)\sin 2(\vec{k}\cdot\vec{x}-\sigma t) + O(k^3a^3) \quad (1)$$

where $U = \left|\vec{U}(U_x, U_y)\right|$ is the current parameter and $k = \left|\vec{k}(k_x, k_y)\right|$ is the wave number whose related vector is normal to the surface undulation front in the wave-current field and lies in the horizontal x-y plane, σ is the angular frequency, a the amplitude of the surface disturbance in the wave-current field, C the celerity, d the mean water depth, t the time, $\vec{x}(x,y)$ the horizontal position vector of a point in the field and z is the vertical axis measured vertically upward from the still water level. The first and second order surface elevation amplitudes are given by a_1 and a_2, respectively. See for example Dean and Dalrymple (1992) for the first order 2D collinear case, and Baddour and Song (1990b) for the second and higher order collinear case.

The relation of the wave number and the angular frequency of the combined wave-current field is given by the following Doppler relation:

$$\sigma - \vec{U}\cdot\vec{k} = \sigma_r \quad (2)$$

where the relative angular frequency in the above equation is described by the following equation:

$$\sigma_r = \sqrt{gk\tanh kd} \quad (3)$$

The dispersion relation for the combined wave-current field is hence:

$$\left(\sigma - \vec{U} \cdot \vec{k}\right) = \sqrt{gk \tanh kd} \tag{4}$$

The periodic free surface elevation η is to first order in amplitude a expressed as:

$$\eta = a\cos(\vec{k} \cdot \vec{x} - \sigma t) + O(a^2) \tag{5}$$

The particle velocity components in the x, y and z direction in the combined wave-current field are obtained as:

$$u_x = U_x + \frac{a\sigma_r}{\sinh kd}\frac{k_x}{k}\cosh k(d+z)\cos(\vec{k} \cdot \vec{x} - \sigma t) + O(k^2a^2) \tag{6}$$

$$u_y = U_y + \frac{a\sigma_r}{\sinh kd}\frac{k_y}{k}\cosh k(d+z)\cos(\vec{k} \cdot \vec{x} - \sigma t) + O(k^2a^2)$$

$$u_z = \frac{a\sigma_r}{\sinh kd}\sinh k(d+z)\sin(\vec{k} \cdot \vec{x} - \sigma t) + O(k^2a^2) \tag{8}$$

The corresponding acceleration components in the x, y and z direction are:

$$a_x = \frac{\sigma a k_x}{k}\frac{\sigma_r}{\sinh kd}\cosh k(d+z)\sin(\vec{k} \cdot \vec{x} - \sigma t) + O(k^2a^2) \tag{9}$$

$$a_y = \frac{\sigma a k_y}{k}\frac{\sigma_r}{\sinh kd}\cosh k(d+z)\sin(\vec{k} \cdot \vec{x} - \sigma t) + O(k^2a^2) \tag{10}$$

$$a_z = -\sigma a\frac{\sigma_r}{\sinh kd}\sinh k(d+z)\cos(\vec{k} \cdot \vec{x} - \sigma t) + O(k^2a^2) \tag{11}$$

The pressure distribution in the wave-current field to second order is obtained from the dynamic free surface boundary condition as:

$$P = -\rho gz - \frac{\rho g a^2 k}{2\sinh 2kd}\left[\cosh 2k(d+z) - 1\right] + \rho g a\frac{\cosh k(d+z)}{\cosh kd}\cos(\vec{k} \cdot \vec{x} - \sigma t)$$
$$+ \frac{3\rho g a^2 k}{2\sinh 2kd}\left[\frac{\cosh 2k(d+z)}{\sinh^2 kd}\right] - \frac{\rho g a^2 k}{2\sinh 2kd} \tag{12}$$

Particle Trajectory

Fig. 2 shows the orientation of the coordinate system for the wave-current combined field where the z-axis is perpendicular at the intersection of the x and y-axes. In the figure, the plane X-Z represents the plane normal to the wave-current field front.

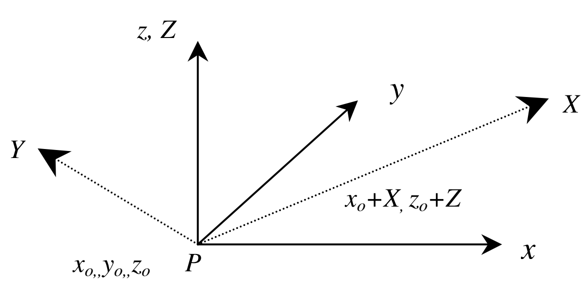

Fig. 2 Axes orientation in 3D for particle trajectory computation

The particle trajectory in the combined wave-current flow is given by the following elliptical expression at any arbitrary point x_o, y_o and z_o. The trajectory in X-Z coordinates is then:

$$\frac{(X-Ut)^2}{\left(\dfrac{ak_x\sigma_r}{\sigma k\sinh kh}\cosh k(d+z)\right)^2}+\frac{Z^2}{\left(\dfrac{a\sigma_r}{\sigma\sinh kh}\sinh k(d+z)\right)^2}=1 \tag{13}$$

where for any value of Y, it is assumed that a water particle moves from its old position (X_o, Y_o, Z_o) to a newer position (X_o+X, Y_o, Z_o+Z).

Figs 3a to 5b show the particle trajectory at the surface (*Z/d=0*) and at the mid-water depth (*Z/d=0.5*) for wave only, wave and current in the same direction and wave with opposing current. In these computations we have used a wave with period T is 4*s*, wave height H is 0.1*m* and water depth d is 10*m*. Figs. 3a and 3b represent the water particle path for wave along $x_oy_oz_o$-XZ vertical plane with *Uo/Co=0*, Figs. 4a and 4b describe the path of the water particle in the same plane with *Uo/Co=0.005* and Figs. 5a and 5b show the particle trajectory when *Uo/Co=-0.005*.

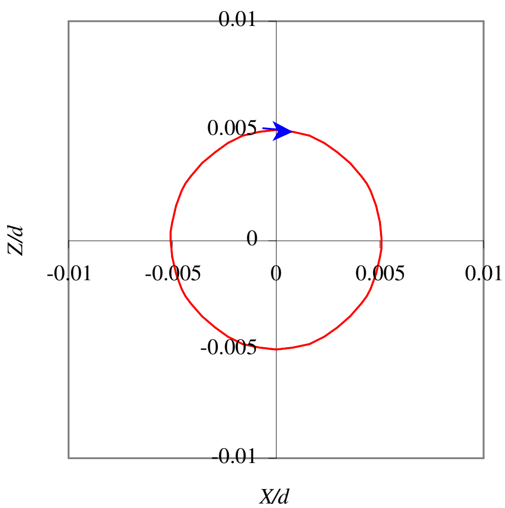

Fig. 3a Particle trajectory at Z/d=0

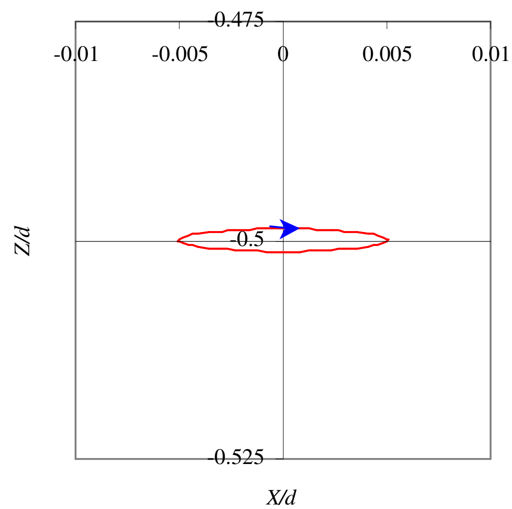

Fig. 3b Particle trajectory at Z/d=-0.5

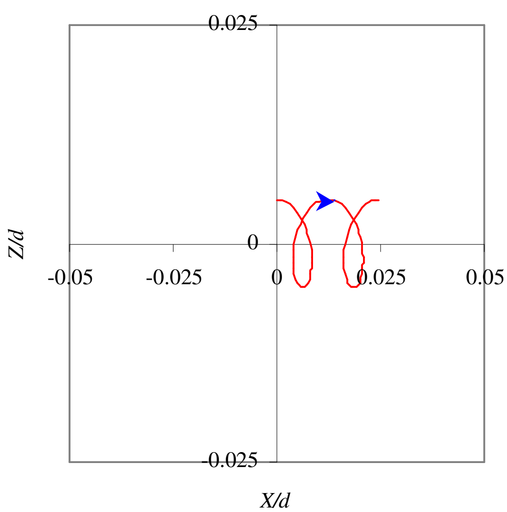

Fig. 4a Particle trajectory at Z/d=0

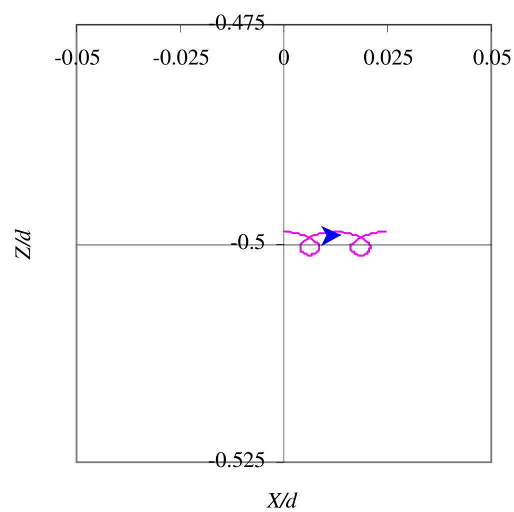

Fig. 4b Particle trajectory at Z/d=-0.5

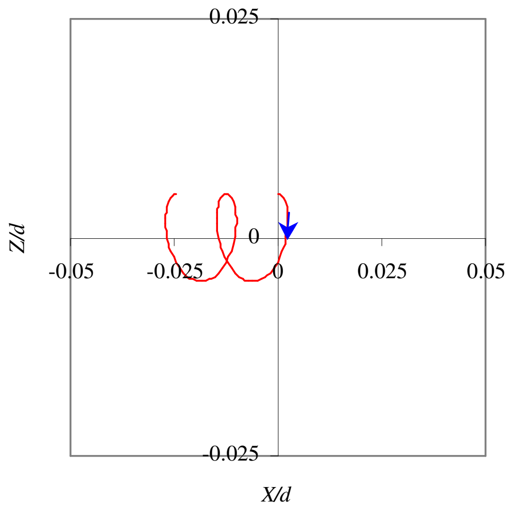

Fig. 5a Particle trajectory at Z/d=0

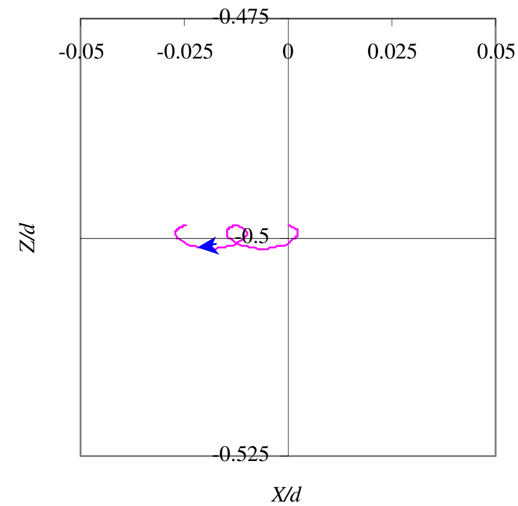

Fig. 5b Particle trajectory at Z/d=-0.5

Equations for Mass, Momentum and Energy Flux

We can obtain the mass flux of the combined wave-current field along the *X-Z* vertical plane in the direction ***N*** through the following relation up to second order in amplitude *a*:

$$\vec{Q}_{wc} = \rho d\vec{U} + \frac{\rho a^2}{2}\vec{k}\left(C - \frac{\vec{U}\cdot\vec{k}}{k}\right)\coth kd + O(k^3a^3) \tag{14}$$

The corresponding momentum flux of the combined wave-current field along the same *X-Z* plane is given as follows:

$$\vec{M}_{wc} = \frac{1}{2}\rho g a^2\left(\frac{1}{2} + \frac{2kd}{\sinh 2kd} + \frac{2\vec{U}\cdot\vec{k}}{\sigma_r}\right) + \frac{1}{2}\rho g d^2\left(1 + \frac{2\left|\vec{U}\right|^2}{gd}\right) + O(k^3a^3) \tag{15}$$

In a similar fashion the net energy flux of the combined wave-current field in the direction of flow in the *X-Z* plane is expressed as:

$$\vec{E}_{wc} = \frac{\rho g\vec{U}}{2}a^2 + \frac{\rho\vec{U}d}{2}\left[\left|\vec{U}\right|^2 + \frac{gk}{\sinh 2kd}a^2\right] + \frac{\rho g a^2}{4}\left[1 + \frac{2kd}{\sinh 2kd}\right]\left[C_r + \frac{\vec{U}\cdot\vec{k}}{k}\right]\frac{\vec{k}}{k}$$

$$+\frac{\rho g a^2}{4\sigma_r}\left[2\vec{U}(\vec{U}\cdot\vec{k}) + \vec{k}\left|\vec{U}\right|^2\right] + O(k^3a^3) \tag{16}$$

Computation of Wave-Current Parameters

Taking the time averages of the flux parameters of the current-free wave field, wave-free current field and wave-current field we pose the following two sets of conservation equations for mass, momentum and energy flux in the $\vec{S}$ and $\vec{N}$ directions, respectively:

$$Q_w \vec{N}_w \cdot \vec{S} + Q_c \vec{N}_c \cdot \vec{S} = 0 \tag{17}$$

$$M_w \vec{N}_w \cdot \vec{S} + M_c \vec{N}_c \cdot \vec{S} = 0 \tag{18}$$

$$E_w \vec{N}_w \cdot \vec{S} + E_c \vec{N}_c \cdot \vec{S} = 0 \tag{19}$$

$$Q_w \vec{N}_w \cdot \vec{N} + Q_c \vec{N}_c \cdot \vec{N} = Q_{wc} \vec{N} \cdot \vec{N} \tag{20}$$

$$M_w \vec{N}_w \cdot \vec{N} + M_c \vec{N}_c \cdot \vec{N} = M_{wc} \vec{N} \cdot \vec{N} \tag{21}$$

$$E_w \vec{N}_w \cdot \vec{N} + E_c \vec{N}_c \cdot \vec{N} = E_{wc} \vec{N} \cdot \vec{N} \tag{22}$$

where subscripts w, c and wc in the above equations, stand for the quantities in the pre-interaction current-free wave field, wave-free current field and in the post-interaction wave-current field, respectively. $\vec{N}_w$ and $\vec{N}_c$ are the given wave and current directions; $\vec{N}$ is the final direction of the combined wave-current field and $\vec{S}$ is the direction normal to $\vec{N}$.

Equations (17) to (19) and Equations (20) to (22) along with Equation (4) are the required set of equations for the evaluation of the properties of the combined wave-current field that evolves when a current-free wave and a wave-free current interact. The iterative solution of Equations (17) to (19) will give the direction of the resultant wave-current field. Once the direction of the combined flow is computed then the system of nonlinear equations (4) and (20) to (22) are solved iteratively for the estimation of the unknowns, a, k, d and U. In this study a Newton iterative method has been utilized. For a given wave with parameters a_o, $k_{o,}$ d_o and current velocity U_o, the computation of the parameters a, k, d and U of the combined wave-current field are obtained from the above equations with a suitable initial guess of the unknowns.

Computational Environment

Maple-V (Rel:4, 1991) is a symbolic programming language using Windows environment. It is used for implementing Newton's algorithm for the numerical solution of the conservation equations together with the dispersion relation. Maple is a system for mathematical computations that can handle symbolic, numeric and graphical procedures in a simplified way. Maple is easily adaptable for those who have experience in other programming computer languages. An important property of Maple is that all the algebraic routine operations in the system are implemented using high-level user language. The basic system, or kernel, is sufficiently compact and efficient to be practical for use in a shared environment or on a personal computer with as little as 2MB of RAM. One of the advantages of Maple is that the user can see an equation in its expanded mathematical format on the monitor while it is taking part in the computations.

An Example

As an application of the established numerical model, it is assumed that a monochromatic surface wave with wave number $k_o = 1.2565539$, wave steepness $H_o/L_o = 0.05$ and relative water depth $d_o/L_o = 1.0$ interacts with normalized current U_o/C_o varying over the range of –0.2 to 0.6. In this application we have also assumed that the wave enters the computational domain at an angle of 10^o degree and while the current is at an angle of 15^o with the positive direction of the x-axis. In the computation wave and current are in same direction (may be with different angles) unless the current U_o represents a negative quantity.

Fig. 6, 7, 8 and 9, respectively shows the variation of the surface disturbance heights, lengths, current like parameters and variation of the direction of the combined wave-current field for the above given conditions. From Figs. 6 and 7, it is observed that wave with a following current will reduce the resulted surface disturbance height and elongate its length and, exactly the opposite incidents take place when current is in the opposite direction of wave.

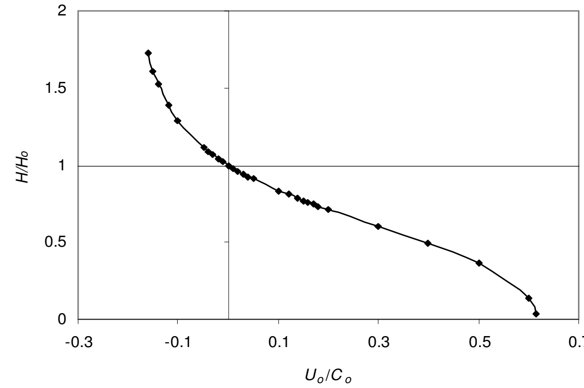

Fig. 6 Variation of wave height H/H_o with U_o/C_o

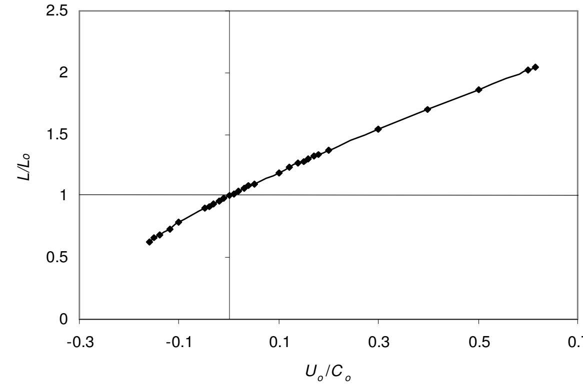

Fig. 7 Variation of wavelength L/L_o with U_o/C_o

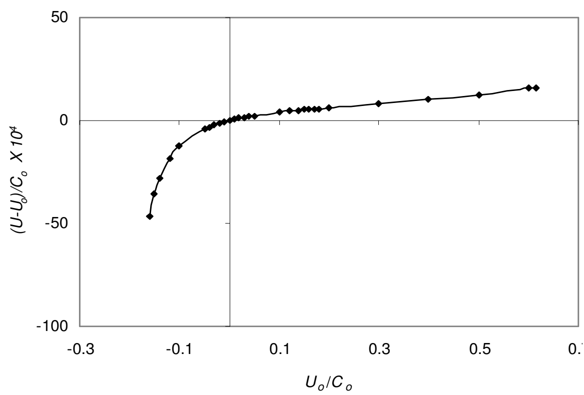

Fig. 8 Variation of current with U_o/C_o

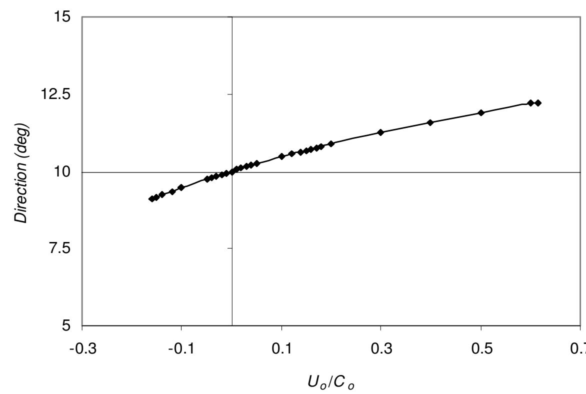

Fig. 9 Variation of direction with U_o/C_o

Fig. 8 shows that the current-like parameter increases when the wave moves with a following current and decreases when they are in opposite directions. In contrast, Fig. 9 shows that the change in the direction of the combined field is much more significant in the case of an opposing current. Figs. 6 to 9 respectively, shows the disturbance height, length, variation of current-like parameters and directions in the combined wave-current field for different current conditions. As expected the surface disturbance height increases with an opposite current and reduces on a following current (Fig. 6) and a reverse behavior is observed for the surface undulation length (Fig. 7). An increase of the current-like parameter is observed for the case of a following current due to wave-induced drift and a reverse behavior is observed for the opposite case (Fig. 8). The variation of the combined wave-current field direction commonly depends on the pre-interaction relative direction of the current-free wave and current fields and magnitude of the current velocity (Fig. 9).

Conclusion

Interaction of a current-free long-crested wave and a wave-free current in a 3D wave-current field for irrotational flow conditions has been formulated in terms of conservation of mass, momentum and energy flux and a dispersion relation. Figs. 3a to 5b show the characteristic properties of the particle trajectory with- and without current along an arbitrary direction in the horizontal *x-y* plane.

Equations (17) to (22) produce the governing conservation equations when Equations (14), (15) and (16) are used to formulate the unknown quantities for the cases of wave, current and wave-current conditions. The obtained equations are used for the numerical computation of the combined field parameters. Maple software environment is used for the iterative solution of the nonlinear system of conservation equations and free-surface dispersion relation. In the computations Equations (17) to (19) are used to find the direction of the combined wave-current field while Equations (20) to (22) together with equation (4) are utilized for the computation of the after-interaction surface disturbance height *H*, its length *L*, mean water depth *d*, current like parameter *U*, and the variation of the direction of the combined wave-current field. Details of the equations and analysis will be presented elsewhere. An example on the application of the present model has been provided.

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