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Hurd, R. A.

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AN APERTURE SYNTHESIS STUDY

– R.A. Hurd –

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OTTAWA

JANUARY 1973

The antenna whose whole pattern fits
To an ideal (and not just in bits),
Is doomed to a failure,
Despite Dolph and Taylor,
Lagrange or Rayleigh and Ritz.

ABSTRACT

A number of antenna synthesis problems involving non-planar apertures have been solved explicitly. In each case the problem contains a constraint in the form of a limitation on the size of the aperture 'Q'.

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AN APERTURE SYNTHESIS STUDY

– R.A. Hurd –

I INTRODUCTION

1.1 Types of synthesis problem

The problem of designing an antenna or an array of antennas to give an approximation to a prescribed radiation pattern is known as antenna synthesis. Compared to analysis, which is the derivation of the characteristics of a given antenna configuration, synthesis is normally the more difficult task. But owing to its greater importance in engineering, it has been worked on considerably in the past. No attempt is made here to document the problem; the interested reader could refer to Chapter 7 of the book by Collin and Zucker (1969) for more details. Despite a large literature, much work remains to be done. In part this is because of inherent difficulty; it is also due to the large range of possible problems, of which we shall now try to give some idea.

Perhaps the most basic synthesis problem is the following: given a linear array of dipoles of fixed spacing and orientation, to find the driving voltages such that the best approximation to a prescribed radiation pattern is obtained. Another problem of greater difficulty arises if the total number of dipoles is fixed, but we ask for their best positions as well as the driving voltages. Problems of a different kind are those in which only a part of the pattern (say the main beam) is specified, with perhaps maximum sidelobes given. Still another variation is to specify only the magnitude of the pattern. This is often all that is required, but it generally leads to problems of greater difficulty. Other variants can be envisaged. There are different error criteria: mean square match of derived and actual patterns, equal ripple approximations, maximum ripple specified, etc. Constraints of various sorts can also be introduced. For instance the magnitudes of the driving voltages could be restricted; in fact, this is often necessary if practical antennas are to be realized. Finally, rather than dealing with discrete arrays, one could try to synthesize continuous distributions, which are sometimes easier to treat.

In short, an almost endless variety of problems can be thought of. It is hardly necessary to say that many will be very difficult to solve.

1.2 Scope of the present work

In this report we shall deal mostly with continuous distributions. This implies the presence of an aperture, which will usually be assumed non-planar. Such apertures might occur on the surface of a satellite, for instance. In addition, we shall normally assume that a constraint in the form of an aperture 'Q' exists. This will be discussed in more detail in Section (1.5). For the present, we merely observe that something of this sort is necessary to avoid 'super-gaining'.

We shall assume that the amplitude and phase of the desired pattern is given, and seek an aperture distribution giving the best-mean-square fit pattern to this. For a number of these problems we are able to give an exact solution.

1.3 Exact and implicit solutions

By an exact solution we mean an answer A in the symbolic form

$$A = f(\hat{P}) \tag{1.2;1}$$

where $\hat{P}$ is the desired pattern and f a known function. To date, very few synthesis problems with constraints have been solved exactly in this sense. The only ones known to the author are by Rhodes (1963, 1972) and Fante (1970). Usually the answer turns up in the implicit form

$$g(A) = \hat{P} \tag{1.2;2}$$

where the known function g remains to be inverted. While this is sometimes feasible by computer, an answer in the form (1) is generally much more desirable.

1.4 Theoretical possibility of synthesis

Before considering complicated synthesis problems with constraints, it is instructive to examine a general unconstrained synthesis problem to see how the aperture field can be determined. Suppose we have an arbitrary 3-dimensional aperture denoted by A in Fig. 1.

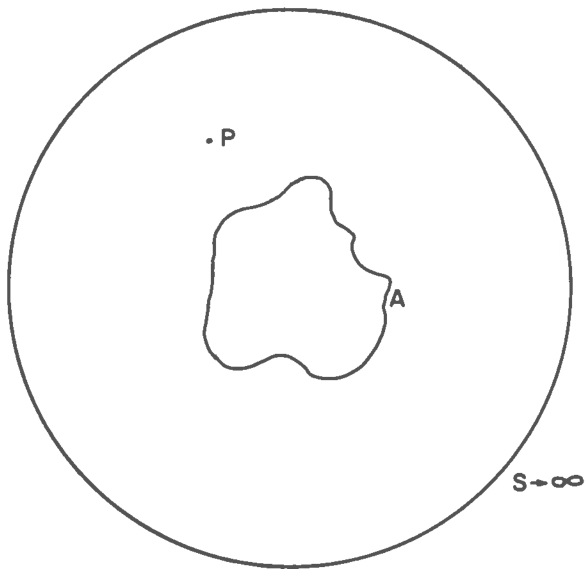


Fig. 1 Arbitrary aperture antenna A

Surround it by a sphere S whose radius is allowed to become infinite. On S we suppose that a scalar field u is known. This will have the form $u = \frac{e^{ikr}}{kr}\hat{P}(\theta,\phi)$ and we suppose that $\hat{P}(\theta,\phi)$ is the desired pattern function. Then, if G is an outgoing wave function singular at a point P outside A and satisfying $G = 0$ on A, we have by Green's theorem:

$$4\pi u(P) = \int_S \left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}\right)ds + \int_A u\frac{\partial G}{\partial n}\,ds\,. \qquad (1.4;1)$$

The integral on S vanishes, so that (1) expresses the field u in terms of its values in the aperture. If now we let $kr \to \infty$ we obtain an integral equation for the unknown aperture distribution function u. It can be proved that a solution to this exists under quite general conditions.

Some points to be observed are:

(a) An analogous treatment exists for vector fields;
(b) the solution u occurs in the implicit form (1.2;2);
(c) the solution u provides an exact fit to the far field $\hat{P}(\theta,\phi)$. If constraints are added, an exact fit is no longer possible in general. In this case a best-approximation pattern must be sought.

1.5 A discussion of constraints

The problems with which we are about to deal will always embody a constraint on the amount of reactive power in the aperture. This is necessary to avoid super-gaining. Highly super-gained antennas are characterized by large and rapid variations of field in the aperture. Such antennas are hard to build and, if built, intolerant to frequency changes.

The degree of super-gain is usually indicated by the super-gain ratio γ, first introduced by Taylor (1955) strictly for line sources. Approximately $\gamma = 1 + |S_i|/S_r$ where S_r and S_i are the real and imaginary parts of the power, and this relation allows the super-gain concept to be extended to aperture antennas.

Clearly $\gamma \geqslant 1$, and large values of γ imply impossible antennas. Evidently, appreciable amounts of reactive power mean poor pattern stability and low realizability. This is reminiscent of resonant circuit theory where Q – the ratio of magnetic or electric stored energy to dissipation – is indicative of the width of the resonance curve. Clearly S_i contains both types of reactive power in the form $\int [\,|H|^2 - |E|^2\,]dv$. Thus $|S_i|/S_r$ is not truly representative of Q, since S_i can be small even when the electric and magnetic powers are large. However, attempts to define a quantity more analogous to Q for apertures have thus far failed. The reason, as shown by Rhodes (1966), is that the integral over all space of either $|H|^2$ or $|E|^2$ separately is divergent, although the difference is finite.

A closely related Q factor was introduced by Uzsoky and Solymar (1957) for arrays of dipoles. Specifically, they put $Q = \Sigma\,|J_i|^2\,/\,S_r$ where J_i is the feeding current of the ith element and the sum is over all elements. Evidently $|J_i|^2$ is representative of real and reactive powers in some sense; so this definition is a possible alternative to S_i/S_r. This concept of Q was extended by Lo, Lee, and Lee (1966) to aperture antennas simply by letting the J_i be the amplitudes of the modes in the aperture. But this definition fails in cases where there is a one-to-one relation between aperture and pattern modes.

Despite its deficiencies we shall normally use $Q = S_i / S_r$ as a constraint. However, Uzsoky's Q is sometimes more convenient, and has been used in one problem, (sec. 8.1).

II RHODES' SYNTHESIS OF A LINE SOURCE

2.1 An outline of Rhodes' solution

One of the very few synthesis problems to be solved exactly when there is a constraint is that of finding the continuous line distribution which gives the best mean square fit to a given pattern. This was solved beautifully by Rhodes (1963) who employed the doubly orthogonal property of the spheroidal functions. Since his work serves as a model for much to follow and since it raises some questions and speculations not considered by Rhodes, a brief outline of his analysis is now given.

Let a line source extend along the x axis from $-a/2$ to $a/2$. Put $v = 2x/a$ and suppose the distribution along the line is $A(v)$. With $u = (\pi a/\lambda)\sin\theta$, the far field pattern is given by

$$P(u) = \frac{1}{2\pi} \int_{-1}^{1} A(v)\, e^{iuv}\, dv \quad . \qquad (2.1;1)$$

See Fig. 2. On taking a Fourier transform we have

$$A(v) = \int_{-\infty}^{\infty} P(u)\, e^{-iuv}\, du \quad . \qquad (2.1;2)$$

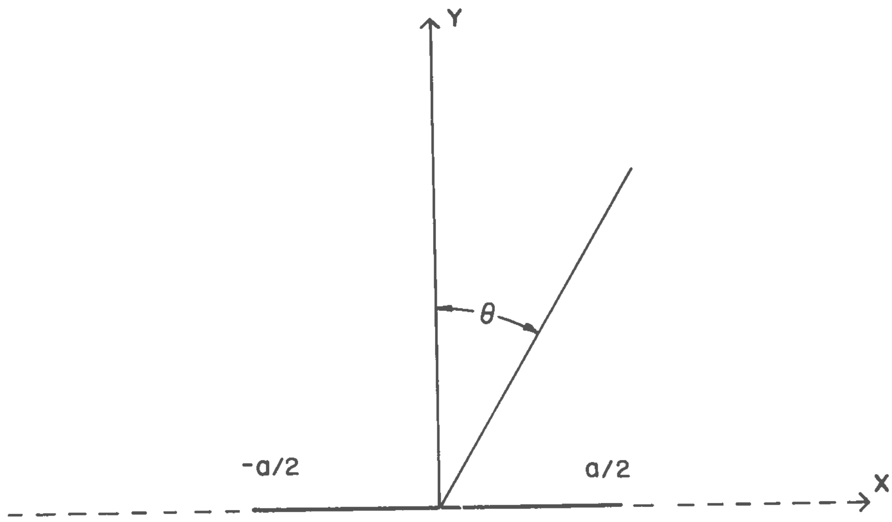


Fig. 2 Line source and coordinate system

The pattern $P(u)$ represents real power for $|u| < \pi a/\lambda$ and reactive power elsewhere. Thus the super-gain ratio can be defined as

$$\gamma = \int_{-\infty}^{\infty} |P(u)|^2 \, du \Big/ \int_{-\pi a/\lambda}^{\pi a/\lambda} |P(u)|^2 \, du \quad . \qquad (2.1;3)$$

Suppose now that $\hat{P}(u)$ is the desired pattern, then the problem faced by Rhodes was to minimize the mean square error

$$\epsilon = \int_{-\pi a/\lambda}^{\pi a/\lambda} |P(u) - \hat{P}(u)|^2 \, du \, ,$$

while holding γ constant. Using the method of Lagrange, this is equivalent to minimizing

$$\epsilon + \mu \left\{ \gamma \int_{-\pi a/\lambda}^{\pi a/\lambda} |P(u)|^2 \, du - \int_{-\infty}^{\infty} |P(u)|^2 \, du \right\} \qquad (2.1;4)$$

where μ is the Lagrange multiplier to be found by substituting the solution in (3).

Rhodes' solution to the problem rests on finding functions orthogonal on two ranges, $(-\infty, \infty)$ and $(-c, c)$, where $c = \pi a/\lambda$. Rhodes recognized that the prolate spheroidal functions $S_{on}(c,u)$ have this property. Thus

$$\int_{-\infty}^{\infty} S_{on}\,(c,u)\, S_{om}\,(c,u)\, du = \kappa_n \delta_{mn} / \lambda_n \; , \qquad (2.1;5)$$

$$\int_{-c}^{c} S_{on}\,(c,\tfrac{u}{c})\, S_{om}\,(c,\tfrac{u}{c})\, du = c\kappa_n \delta_{mn} \quad , \qquad (2.1;6)$$

where κ_n, λ_n are certain normalization constants.

If the given pattern is expanded in terms of the S_{on}, we have

$$\hat{P}(u) = \sum_0^N \hat{a}_n \, S_{on}\,(c,\tfrac{u}{c}) \qquad (2.1;7)$$

with the $\hat{a}_n$ known. Let

$$P(u) = \sum_0^N a_n \, S_{on}\,(c,\tfrac{u}{c}) \quad . \qquad (2.1;8)$$

Then after substitution in (4) and differentiation with respect to the real and imaginary parts of a_n one finds that

$$a_n = \hat{a}_n \, [\, 1 + \mu\, (\lambda_n^{-1} - \gamma)]^{-1} \, , \qquad (2.1;9)$$

while μ is got by solving

$$\sum_0^N \frac{\kappa_n \, |\hat{a}_n|^2}{(\lambda_n^{-1} - \gamma)\, [\mu + (\lambda_n^{-1} - \gamma)^{-1}]^2} = 0 \quad . \qquad (2.1;10)$$

While apparently the complete solution to the problem is given in (9) and (10), the reader might feel that there are still some unanswered questions, as follows. Are (9) and (10) sufficient to ensure a minimum; and if not, what extra restrictions does this entail? Can γ be arbitrarily chosen? As (7) is a best mean square approximation to $\hat{P}(u)$ (with the $\hat{a}_n$ determined by Fourier methods), does the solution (8) represent a best mean square approximation to a best mean square approximation to $\hat{P}(u)$?

We shall clear up these questions in connection with another minimization problem to be considered in Sec. 3.2. Here the analysis involves the much simpler exponential and Bessel functions, as opposed to the fairly complicated spheroidal ones. Our answers will remain valid regardless of the problem.

2.2 An implication of Rhodes' solution

The spheroidal functions of the previous section were introduced because their doubly orthogonal properties allowed a closed form solution to the synthesis problem. Evidently Rhodes did not realize it, but there is a second reason why the spheroidal functions are appropriate to the problem. They are appropriate simply *because they arise naturally when the boundary value problem of a straight line conductor is solved in its natural system of coordinates (the prolate spheroidal system).* Looked at from this new angle, any synthesis problem ought to be exactly solvable if the aperture consists of a complete coordinate surface, and if the particular coordinate system is one for which the wave equation separates; always providing a suitable definition of the super-gain ratio is available.

Since the super-gain ratio is roughly $1 + |S_i|/S_r$ for a line distribution, and the quantity S_i/S_r is easily found by a direct integration of the complex Poynting vector across the aperture (rather than by an integration of the pattern in Rhodes' case), we have a method of introducing a power constraint involving only integrations across the aperture, where the orthogonal properties of the natural functions of the particular coordinate system can be exploited.

We are then led to the main conjecture of the present essay: "A synthesis problem with a constraint on the aperture Q $(= S_i/S_r)$ is exactly solvable if the aperture coincides with a surface for which the corresponding diffraction problem is exactly solvable".

This conjecture turns out *not* to be true; for instance we have been unable to solve the problem of the parabolic cylindrical aperture. However, it does work in many cases, and we have been led thereby to the exact solution of several aperture synthesis problems.

III SYNTHESIS OF A CIRCULAR CYLINDRICAL APERTURE

3.1 Analysis of the problem – E-polarization

In this problem we are given a circular cylindrical aperture of radius a, extending to infinity in the $\pm z$ directions, and we look for a z independent distribution of E_z in the aperture which gives the best mean square fit to a given far field pattern $P(\phi)$.

We regard this problem as the key one of the report since its solution is the simplest of all and not only gives the answers to the questions raised in Sec. 2.1 but points the way to the solution of the more complex problems treated later. Accordingly, it is treated in some detail.

Since only E_z exists in the aperture, there are only 3 field components outside, namely E_z, H_ρ, H_ϕ. Assuming and suppressing a time-dependence $e^{-i\omega t}$, the relevant field components are given by

$$E_z(\rho,\phi) = \sum_{-\infty}^{\infty} A_m e^{im\phi} H_m^{(1)}(k\rho) , \tag{3.1;1}$$

$$H_\phi(\rho,\phi) = \frac{ik}{\omega\mu_0} \cdot \sum_{-\infty}^{\infty} A_m e^{im\phi} H_m^{(1)'}(k\rho) , \tag{3.1;2}$$

the A_m being undetermined constants. From (1), the far field is easily found:

$$E_z(\rho,\phi) = e^{i(k\rho - \pi/4)} (2/\pi k\rho)^{1/2} \sum_{-\infty}^{\infty} A_m e^{-im\pi/2 + im\phi} , \tag{3.1;3}$$

and, with $a_m = A_m e^{-im\pi/2}$, the pattern is

$$P(\phi) = \sum_{-\infty}^{\infty} a_m e^{im\phi} . \tag{3.1;4}$$

The complex power S in the aperture $\rho = a$ is given by $(a/2) \int_0^{2\pi} (\mathbf{E} \times \mathbf{H}^*)_\rho \, d\phi$. On using (1) and (2) to evaluate the integral, we get

$$S = S_r + iS_i = (i\pi ka/\omega\mu_0) \sum_{-\infty}^{\infty} |a_m|^2 H_m^{(1)}(ka) H_m^{(2)'}(ka) . \tag{3.1;5}$$

Then, after introducing the Wronskian relation for the Hankel functions:

$$S_r = (2/\omega\mu_0) \sum_{-\infty}^{\infty} |a_m|^2 , \tag{3.1;6}$$

$$S_i = (\pi ka/\omega\mu_0) \sum_{-\infty}^{\infty} |a_m|^2 Z_m , \tag{3.1;7}$$

with

$$Z_m = J_m(ka) J_m'(ka) + Y_m(ka) Y_m'(ka) . \tag{3.1;8}$$

It follows that

$$Q = \frac{S_i}{S_r} = \frac{\frac{1}{2}\pi ka \sum_{-\infty}^{\infty} |a_m|^2 Z_m}{\sum_{-\infty}^{\infty} |a_m|^2} . \tag{3.1;9}$$

If $\hat{P}(\phi)$ is the pattern we wish to obtain:

$$\hat{P}(\phi) \;=\; \sum_{-\infty}^{\infty} \hat{a}_m \, e^{im\phi} \,, \qquad (3.1;10)$$

with $\hat{a}_m$ found from

$$\hat{a}_m \;=\; \frac{1}{2\pi} \int_0^{2\pi} e^{-im\phi} \, \hat{P}(\phi) \, d\phi \,, \qquad (3.1;11)$$

then the problem is to minimize

$$\epsilon \;=\; \int_0^{2\pi} |P(\phi) \;-\; \hat{P}(\phi)|^2 \, d\phi \qquad (3.1;12)$$

subject to Q = constant. Or, by the Lagrange method, we minimize

$$\int_0^{2\pi} |P(\phi) - \hat{P}(\phi)|^2 \, d\phi \;+\; \mu \sum_{-\infty}^{\infty} (Q - \tfrac{1}{2}\,\pi ka \, Z_m) \; |a_m|^2 \;. \qquad (3.1;13)$$

If (4) and (10) are substituted into (13), the integration performed, and the derivatives with respect to the real and imaginary parts of the a_m set equal to zero, we get

$$a_m \;=\; \frac{2\pi \, \hat{a}_m}{2\pi \,+ \mu \,(Q - \tfrac{1}{2}\pi ka \, Z_m)} \;. \qquad (3.1;14)$$

To find μ, (14) is inserted in (9) yielding

$$\sum_{-\infty}^{\infty} \; |\hat{a}_m|^2 \, . \; |2\pi \,+\, \mu \; (Q \,-\, \tfrac{1}{2}\pi ka \, Z_m)| \, .^{-2} \; (Q \,-\, \tfrac{1}{2}\pi ka \, Z_m) = 0 \;\;. \qquad (3.1;15)$$

This is a transcendental equation to be solved for μ, but it is normally quite a simple job on the computer. It will be observed that (14) and (15) are the analogues of (2.1;9) and (2.1;10), but that they involve much simpler functions.

The solution is now formally complete: substitution of (14) in (4) and (1) gives the realized pattern and the aperture distribution.

The actual minimum error is easily found to be

$$\epsilon \;=\; \frac{\mu^2}{2\pi} \sum_{-\infty}^{\infty} \left| \frac{\hat{a}_m \,(Q - \tfrac{1}{2}\pi ka \, Z_m)}{1 + (\mu/2\pi)(Q - \tfrac{1}{2}\pi ka \, Z_m)} \right|^2 \;. \qquad (3.1;16)$$

3.2 Subsidiary proofs, bounds, discussion

A vital question is: "How does Z_m behave as a function of m? " It is easily answered.

By differentiation of Nicholson's formula (Magnus and Oberhettinger (1949), page 30) we have

$$Z_m = (-8/\pi^2) \int_0^\infty K_1(2ka \sinh t) \sinh t \cosh 2mt \, dt \; , \qquad (3.2;1)$$

where K_1 is the modified Hankel function. Since the integrand is positive, Z_m is negative. Moreover, since $\cosh 2mt$ is increasing with $|m|$, Z_m decreases monotonically with $|m|$. It will be noted that (1) also shows that $Z_{-m} = Z_m$.

We are now able to treat the question raised in Sec. (2.1) as to whether a minimum of error has been achieved. It is well known that a sufficient condition for a minimum of a function $f(x_1, \ldots x_n)$ of n variables is that the Hessian matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$ be positive definite. In this case (and in Rhodes') the off-diagonal terms are zero; so the condition reduces to

$$\partial^2 f / \partial x_i^2 > 0 \; , \quad i = 1,2, \ldots \ldots n, \ldots . \qquad (3.2;2)$$

For the cylindrical aperture, (2) translates into

$$2\pi + \mu(Q - \tfrac{1}{2}\pi ka Z_m) > 0, \quad m = 0,1, \ldots . \qquad (3.2;3)$$

This puts conditions on μ which can only be understood after reference to (3.1;15). Here it is apparent that the factor $Q - \frac{1}{2}\pi ka Z_m$ must change sign as a function of m, if a finite value for μ is to be obtained. Thus we must have

$$Q - \tfrac{1}{2}\pi ka Z_m < 0 \qquad (3.2;4)$$

for all $|m| < M_0$ say. Then from (3)

$$\mu < \frac{2\pi}{\frac{1}{2}\pi ka Z_m - Q} \; , \qquad |m| < M_0 \; ,$$

which, since Z_m is monotonic, can be replaced by

$$\mu < \frac{2\pi}{\frac{1}{2}\pi ka Z_0 - Q} \; . \qquad (3.2;5)$$

As $m \to \infty$, $Z_m \to -\infty$, so that (3) also gives

$$\mu > 0 \; . \qquad (3.2;6)$$

If, as must always occur in practice, our series are finite, then (6) becomes

$$\mu > \frac{2\pi}{\frac{1}{2}\pi ka Z_M - Q} \; , \qquad (3.2;7)$$

where M is the upper limit of the sums. Clearly a solution of (3.1;15) must always be sought which satisfies the two inequalities (5) and (7).

The inequality (4) implies that Q cannot be arbitrarily chosen. It must in fact satisfy

$$\tfrac{1}{2}\pi ka\, Z_M < Q < \tfrac{1}{2}\pi ka\, Z_0 \ . \qquad (3.2;8)$$

We note that Q is always negative. This is just a consequence of our choice of a multiplier in (3.1;9); clearly any positive or negative one would do.

According to (7), negative values of μ are allowed. Then there must exist a value of Q which yields $\mu = 0$, corresponding to zero error. This Q value (called Q_{perf}) can be got from (3.1;15):

$$Q_{\mathrm{perf}} = \tfrac{1}{2}\pi ka \sum_{-M}^{M} |\hat{a}_m|^2 Z_m \Big/ \sum_{-M}^{M} |\hat{a}_m|^2 \ . \qquad (3.2;9)$$

Values of $Q < Q_{\mathrm{perf}}$ lead again to non-zero error, so that the lower bound of (8) should be replaced by Q_{perf}. If M is infinite, $Q_{\mathrm{perf}} = -\infty$ and this is also the lower bound of Q according to (8).

It is implicit in much of the foregoing that μ is real. Doubtless (3.1;15) yields complex μ-roots, but since these cannot satisfy (3), only real roots need be considered. The question now arises whether there is a unique real root of (3.1;15) satisfying (5) and (7). The answer is yes, as can be seen from writing the L.H.S. of (3.1;15) in the following way:

$$\left\{ \sum_{|m| \leq M_0} + \sum_{M_0 < |m| \leq M} \right\} \cdot \left\{ |\hat{a}_m|^2 \cdot |2\pi + \mu(Q - \tfrac{1}{2}\pi ka\, Z_m)|^{-2} \cdot (Q - \tfrac{1}{2}\pi ka\, Z_m) \right\} \qquad (3.2;10)$$

Each term of the first sum is negative, while the second has only positive members. For μ at its minimum value (7), the second term gives $+\infty$ while the first term is maximum but finite. As μ increases, both terms decrease monotonically. The first term tends to $-\infty$ at maximum μ, while the second term is minimum. See Figure 3. It is evident, then, that a μ exists for which the first term is minus the second. In view of the monotonic behaviour, this must be the only root satisfying (3.1;15) in the required range.

Finally we answer the question: is our solution a mean square approximation to a mean square approximation? The answer is *no*, as can be seen by using the exact $\hat{P}(\phi)$ in (3.1;12) rather than its M term approximation.

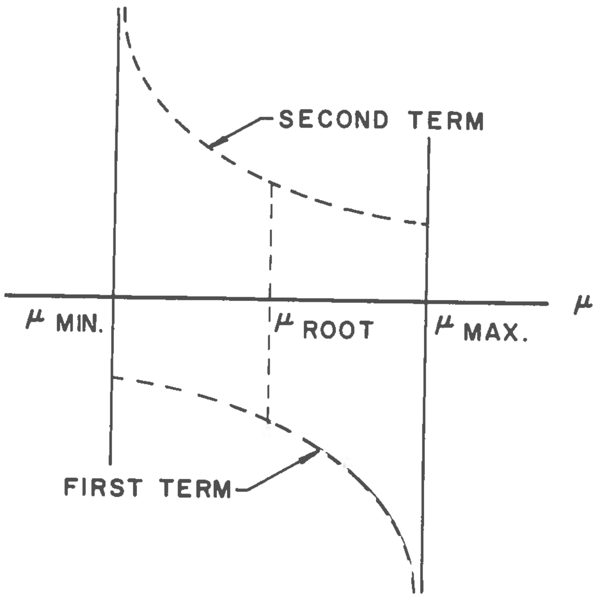


Fig. 3 Graph showing existence of a root μ

3.3 Summary of relevant formulae

The following steps give succinctly the synthesis procedure assuming that a desired pattern $\hat{P}(\phi)$, the radius a of the antenna aperture, and the Q (a negative number) are prescribed.

(a) Set $M \simeq 2ka$.

(b) Calculate $Z_m = J_m(ka)\, J_m'(ka) + Y_m(ka)\, Y_m'(ka)$ for $|m| \leqslant M$.

(c) Calculate $\hat{a}_m = (\frac{1}{2}\pi) \int_0^{2\pi} e^{-im\phi}\, \hat{P}(\phi)\, d\phi, \quad |m| \leqslant M.$

(d) Calculate $Q_{\text{perf}} = \frac{1}{2} ka\pi \sum_{-M}^{M} |\hat{a}_m|^2\, Z_m \Big/ \sum_{-M}^{M} |\hat{a}_m|^2 .$

(e) Check that $Q_{\text{perf}} < Q < \frac{1}{2}\pi ka\, Z_0$. If not, Q must be reassigned.

(f) Calculate μ from $\sum_{-M}^{M} |\hat{a}_m [\pi + \mu\,(Q - \frac{1}{2}\pi ka\, Z_m)]^{-1}|^2 \cdot (Q - \frac{1}{2}\pi ka\, Z_m) = 0.$

It must lie in the range $\dfrac{2\pi}{\frac{1}{2}ka\, Z_m - Q} < \mu < \dfrac{2\pi}{\frac{1}{2}ka\, Z_0 - Q}$.

(g) Calculate $a_m = \dfrac{2\pi\, \hat{a}_m}{2\pi + \mu(Q - \frac{1}{2}\pi ka\, Z_m)}$.

(h) The achieved pattern is given by $P(\phi) = \sum_{-M}^{M} a_m\, e^{im\phi}$.

(i) The aperture distribution is $E_z\,(a, \phi) = \sum_{-M}^{M} e^{im\pi/2}\, a_m\, H_m^{(1)}(ka)\, e^{im\phi}$.

3.4 Numerical results

A computer program was written to calculate realized patterns and the aperture fields. Figures 4–11 give realized patterns for different Q's and an aperture of $ka = 20$, with $M = 29$. The desired pattern is unity for $-45° \leqslant \phi \leqslant 45°$ and zero outside. The pattern marked unconstrained is the 29 term Fourier series approximation to the square wave ideal pattern. Some of the corresponding amplitude distributions are given in Figs. 12–15. One will observe little pattern improvement beyond $Q = .0260$, but extreme sensitivity to Q before this. (In labelling these patterns we have changed Q into $-Q$). Apparently a good compromise choice would be $Q = .02540$.

For comparison, Figs. 16–20 repeat the above but with the beamwidth narrowed to 45°. It is evident that a Q of .02540 no longer gives a good approximate pattern; something greater than about $Q = .026$ seems necessary here. This finding fits with the intuitive idea that higher Q's are needed to approximate more extreme patterns.

3.5 Analysis for H-polarization

This problem is almost identical with the preceding. We have

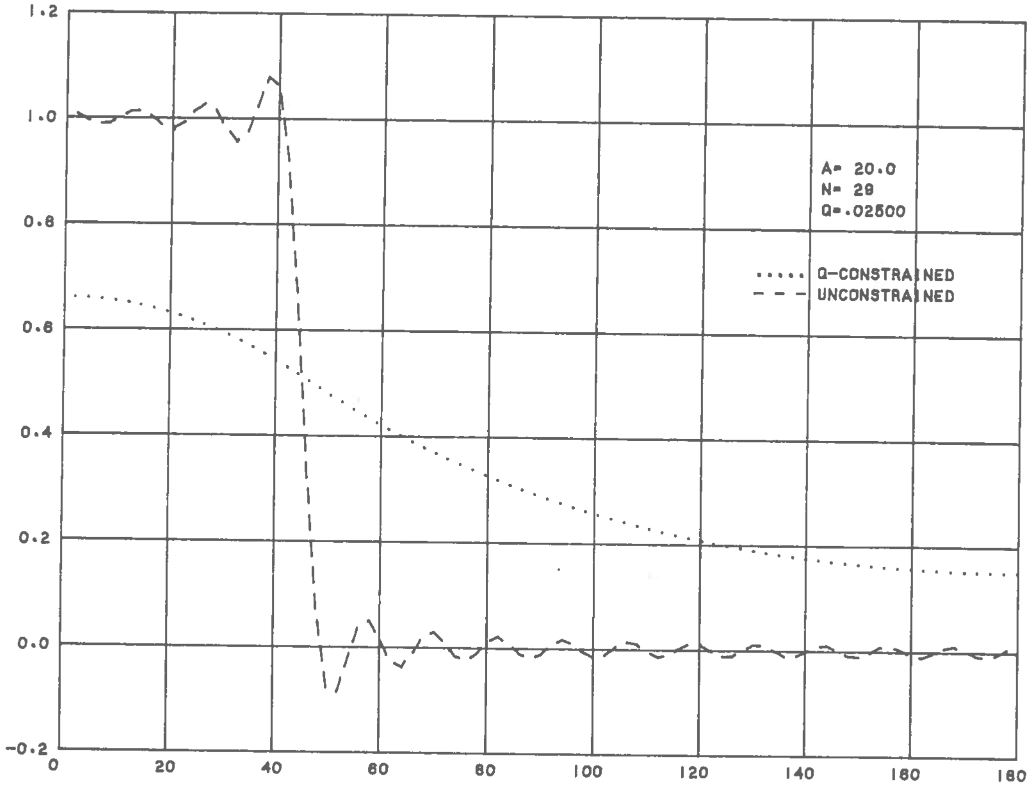
$$H_z\,(\rho, \phi) = \sum_{-\infty}^{\infty} A_m\, e^{im\phi}\, H_m^{(1)}\,(k\rho)\ , \tag{3.5;1}$$

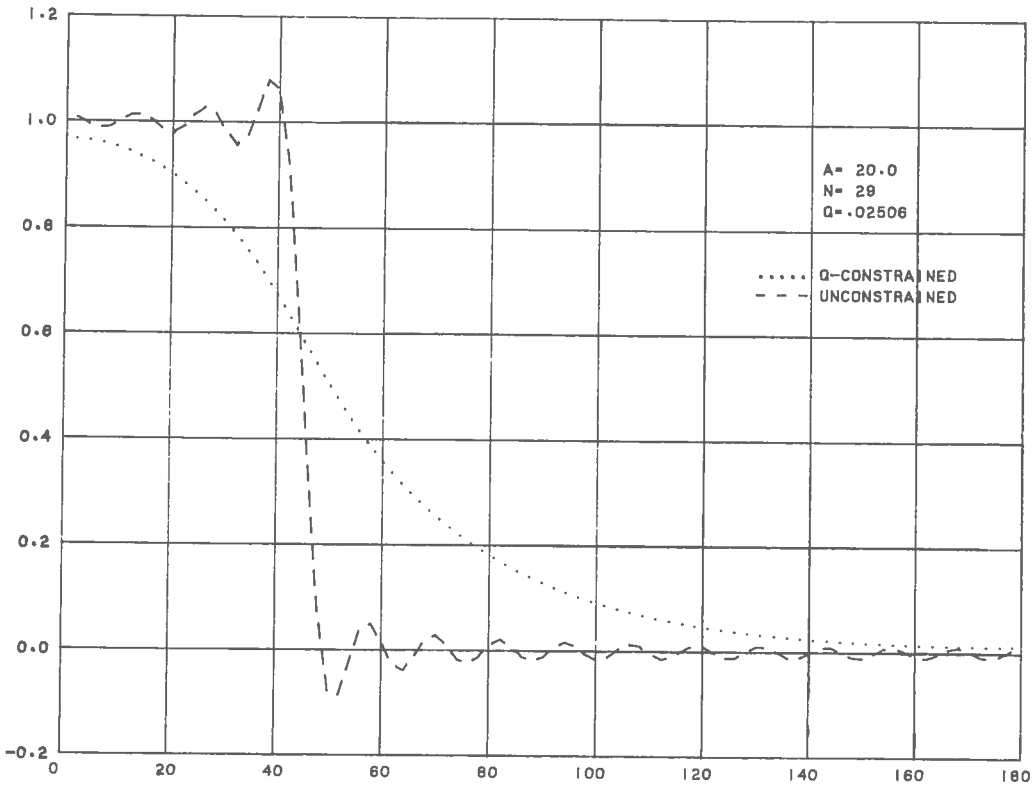
$$E_\phi\,(\rho, \phi) = (-ik/\omega\epsilon_0) \sum_{-\infty}^{\infty} A_m\, e^{im\phi}\, H_m^{(1)\prime}\,(ka)\ . \tag{3.5;2}$$

We now define the desired and achieved patterns in terms of H_z. The power in the aperture becomes

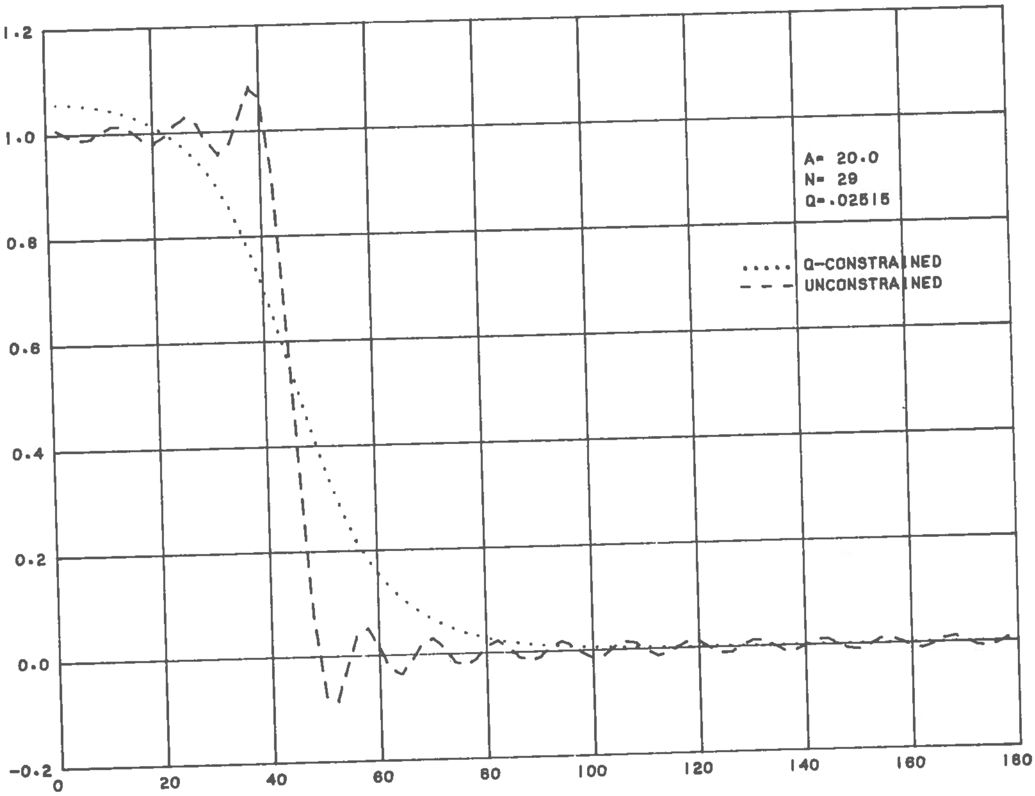
$$S_r + iS_i = (-i\pi ka/\omega\epsilon_0) \sum_{-\infty}^{\infty} |a_m|^2\, H_m^{(1)\prime}\,(ka)\; H_m^{(2)}\,(ka) \tag{3.5;3}$$

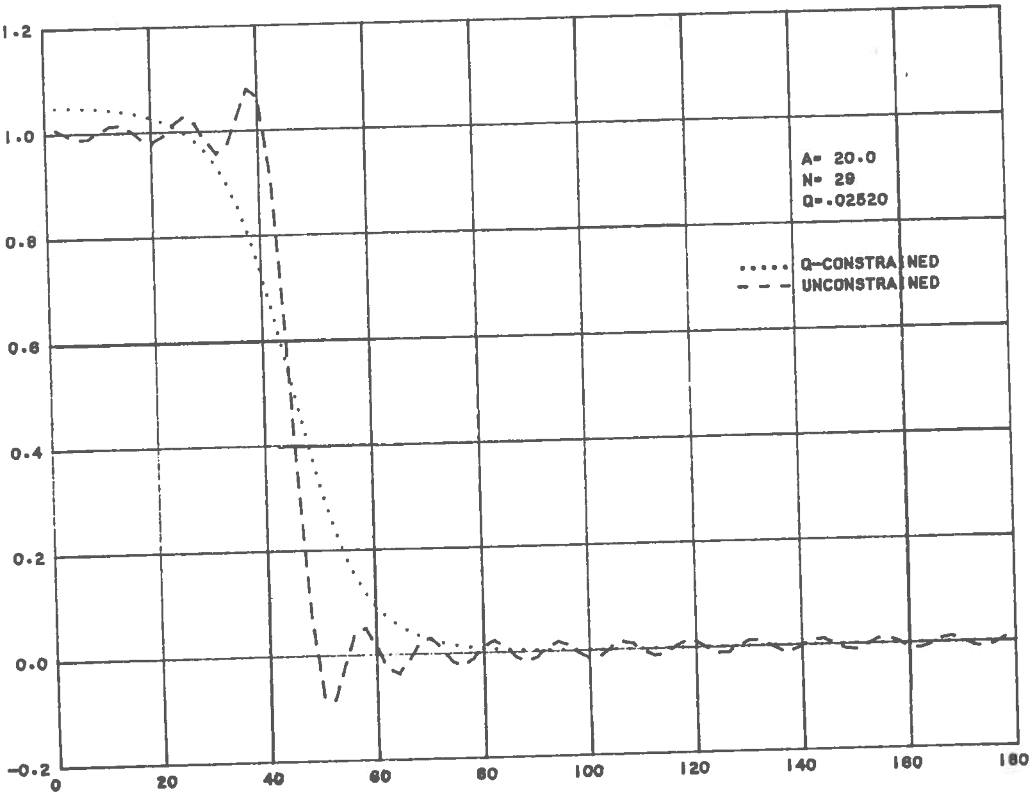
which is μ_0/ϵ_0 times the complex conjugate of (3.1;5). This just has the effect of altering the sign of Q, so that all the previous analysis goes through otherwise unchanged. But we note that the aperture electric field will of course be different.



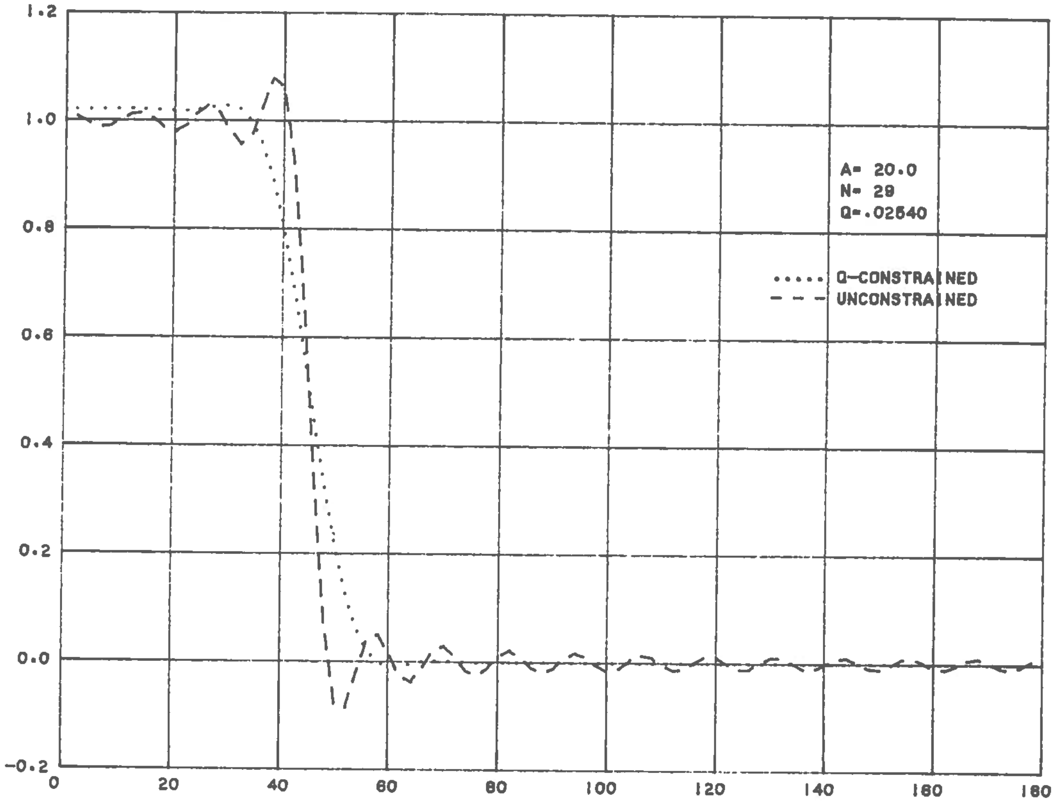


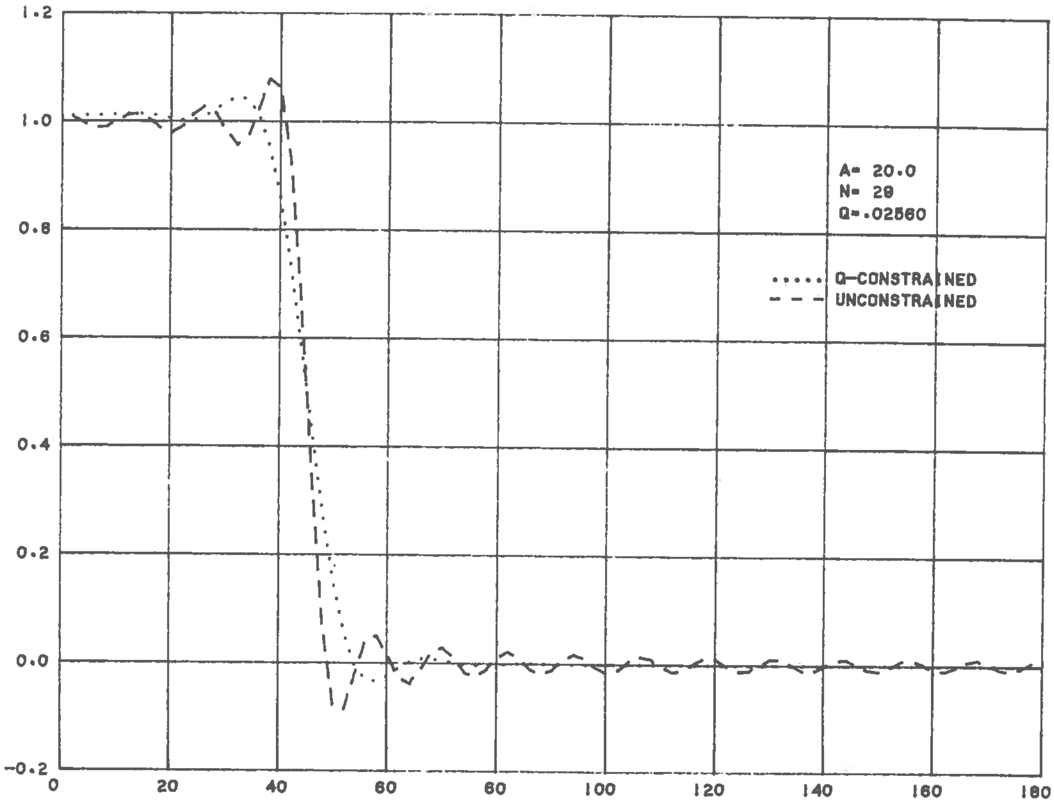
Figs. 4 & 5 Realized patterns of a cylindrical aperture; 90° beam; $Q = .02500$ to $.02506$.



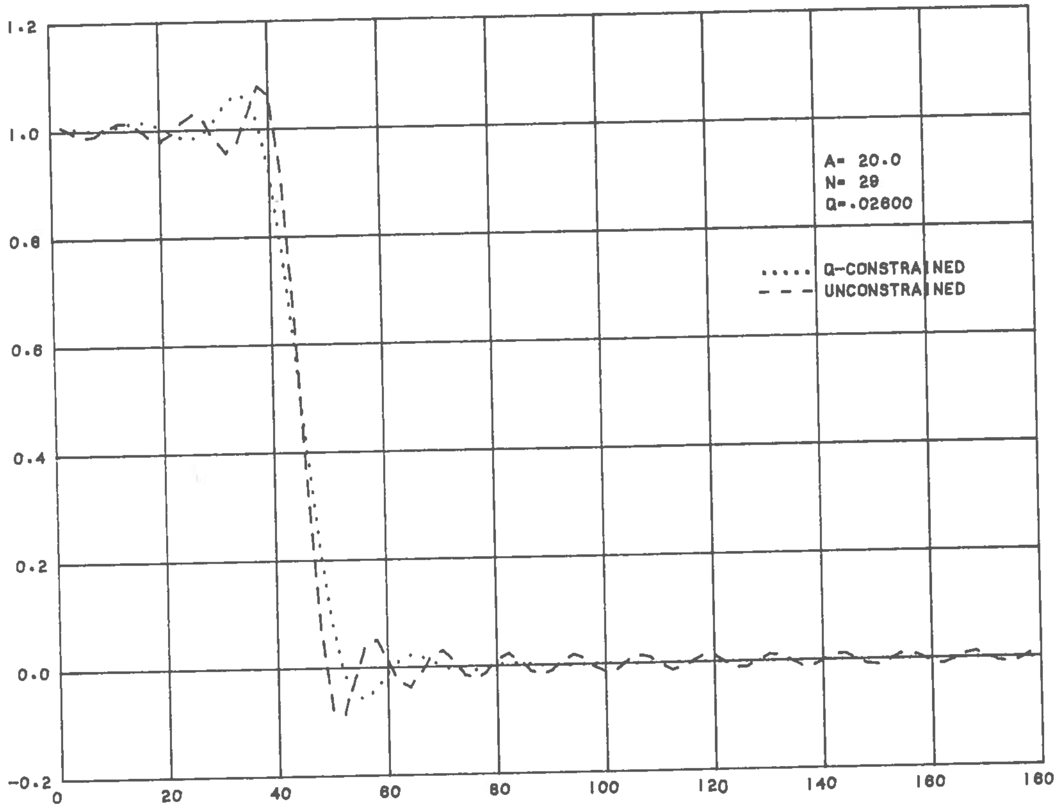


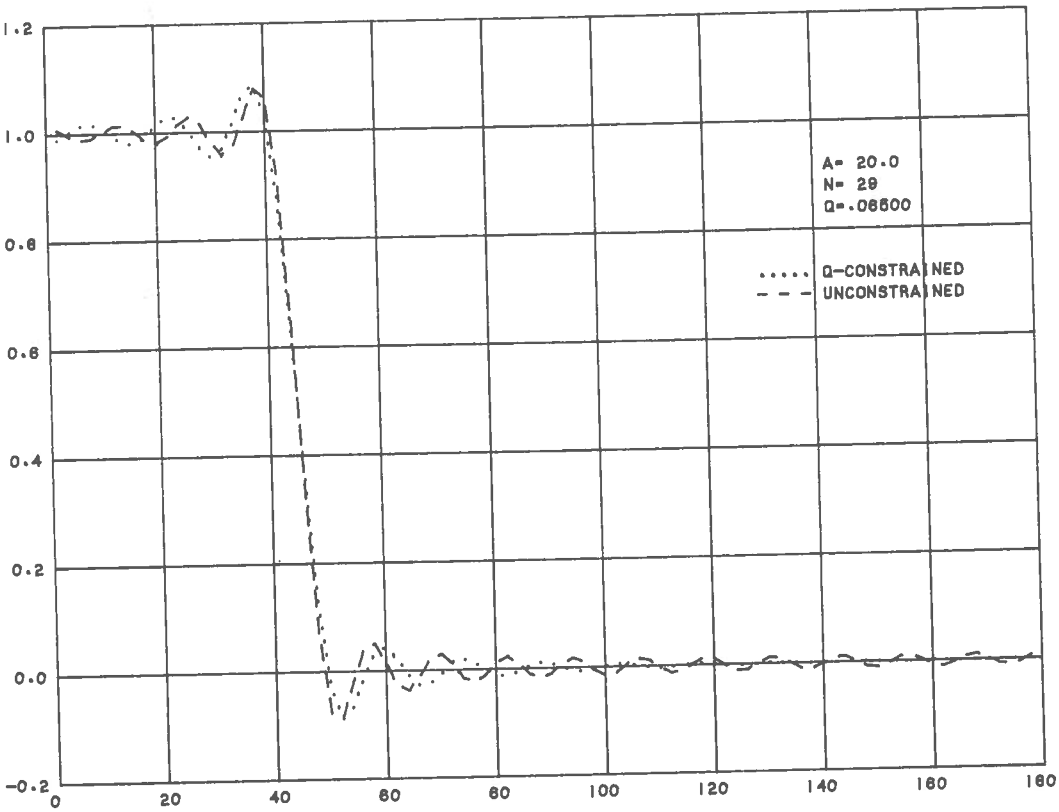
Figs. 6 & 7 Realized patterns of a cylindrical aperture; 90° beam; Q = .02515 to .02520.



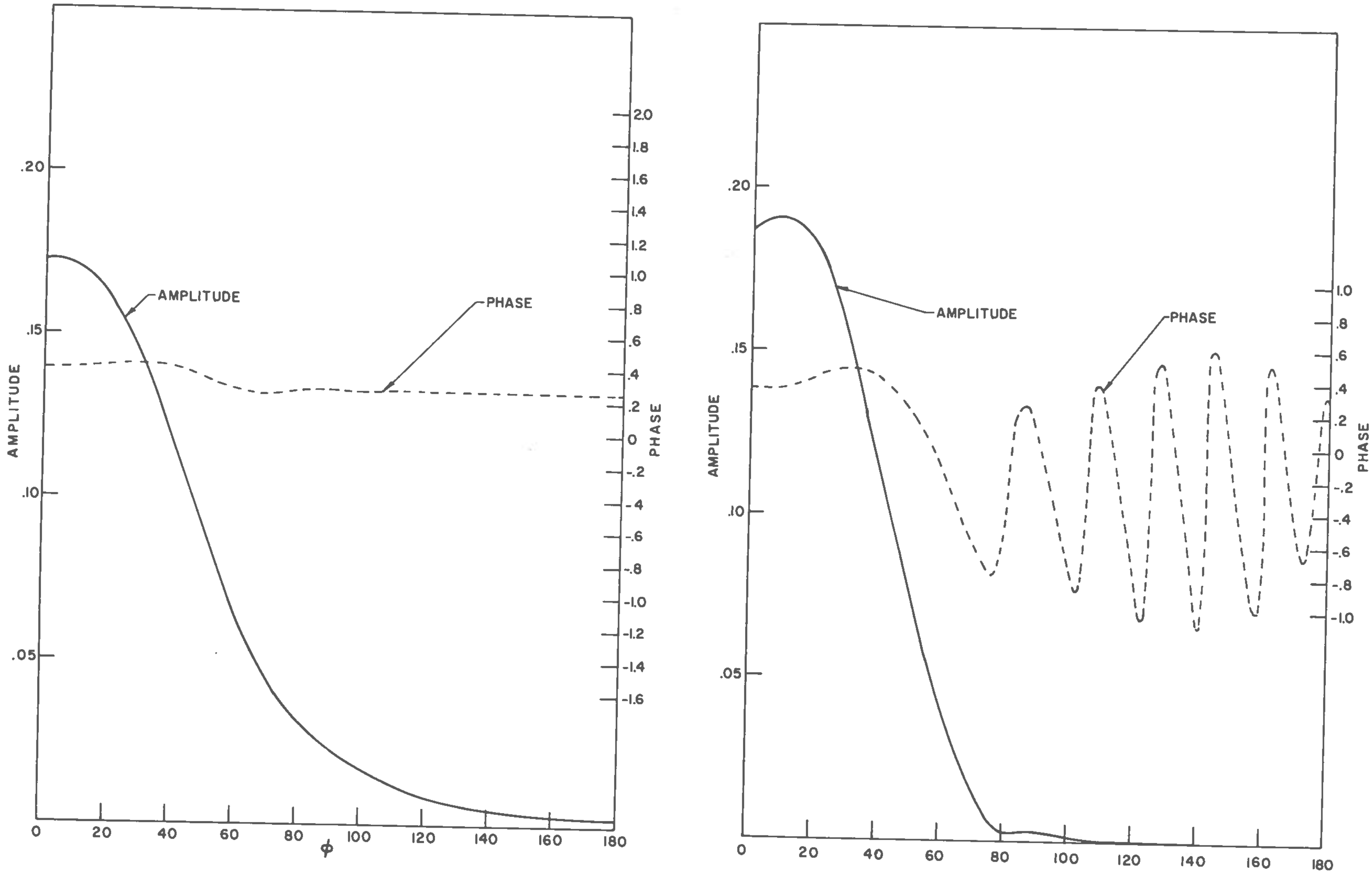


Figs. 8 & 9 Realized patterns of a cylindrical aperture; 90° beam; Q = ;02540 to .02560.

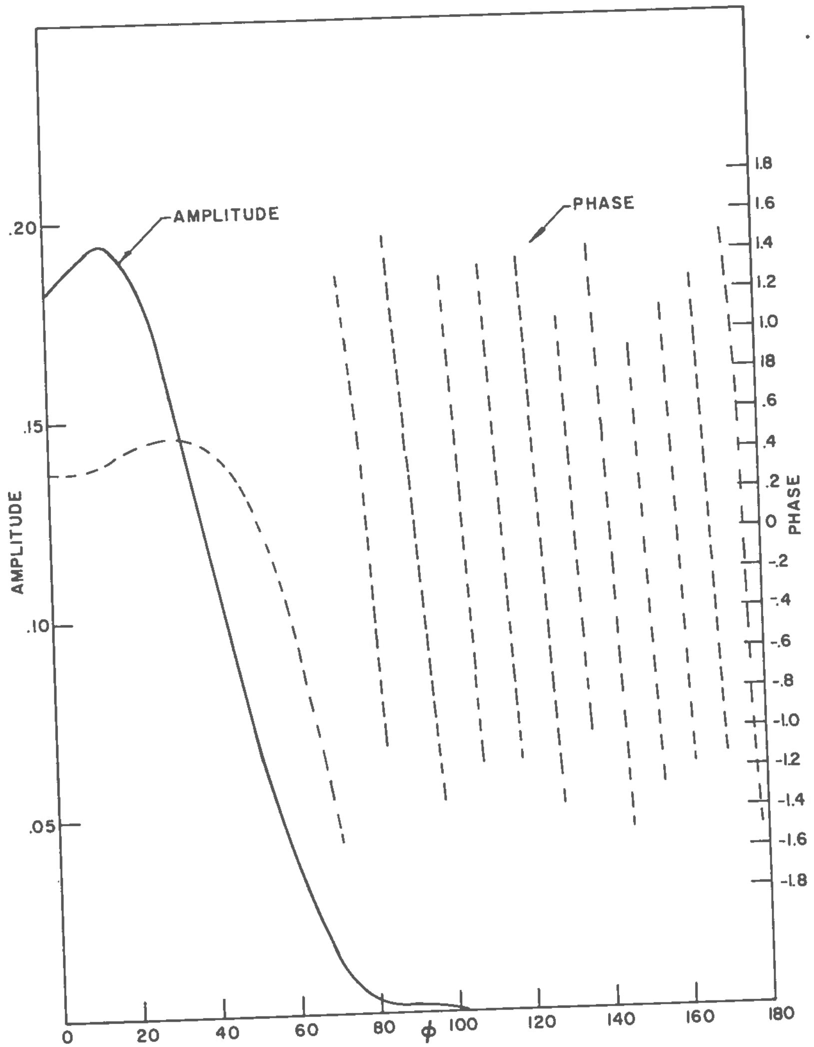


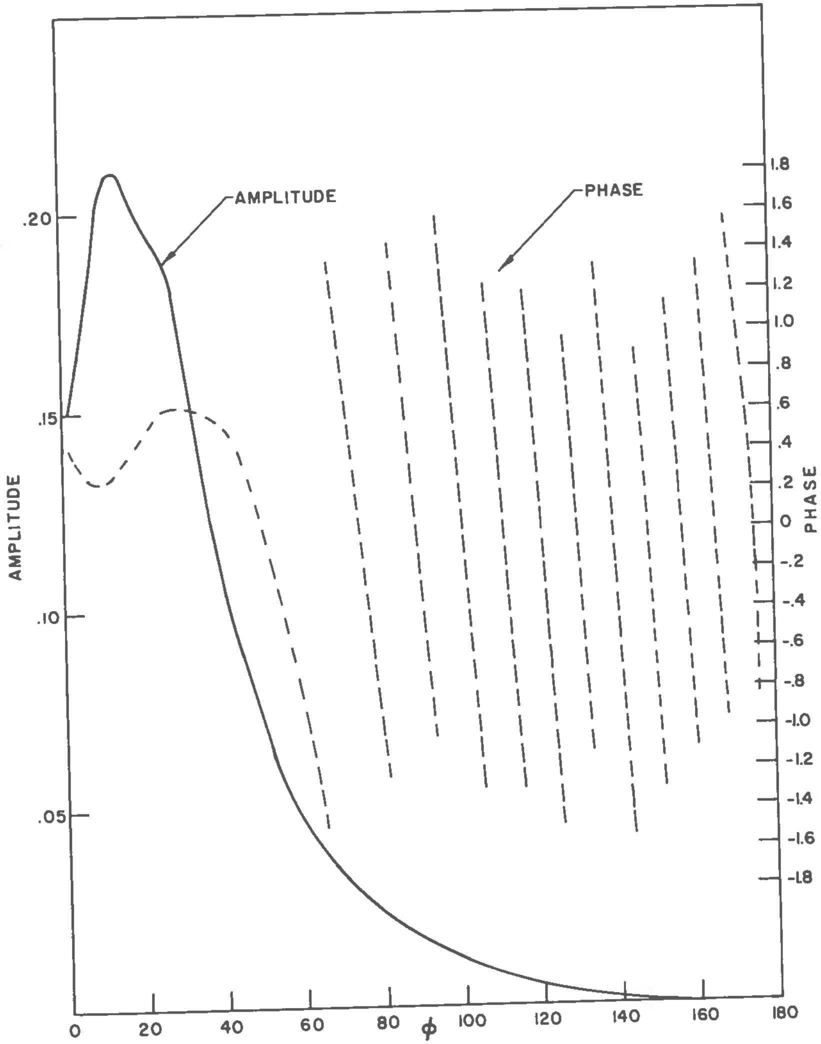


Figs. 10 & 11 Realized patterns of a cylindrical aperture; 90° beam; Q = .02600 to .06500.

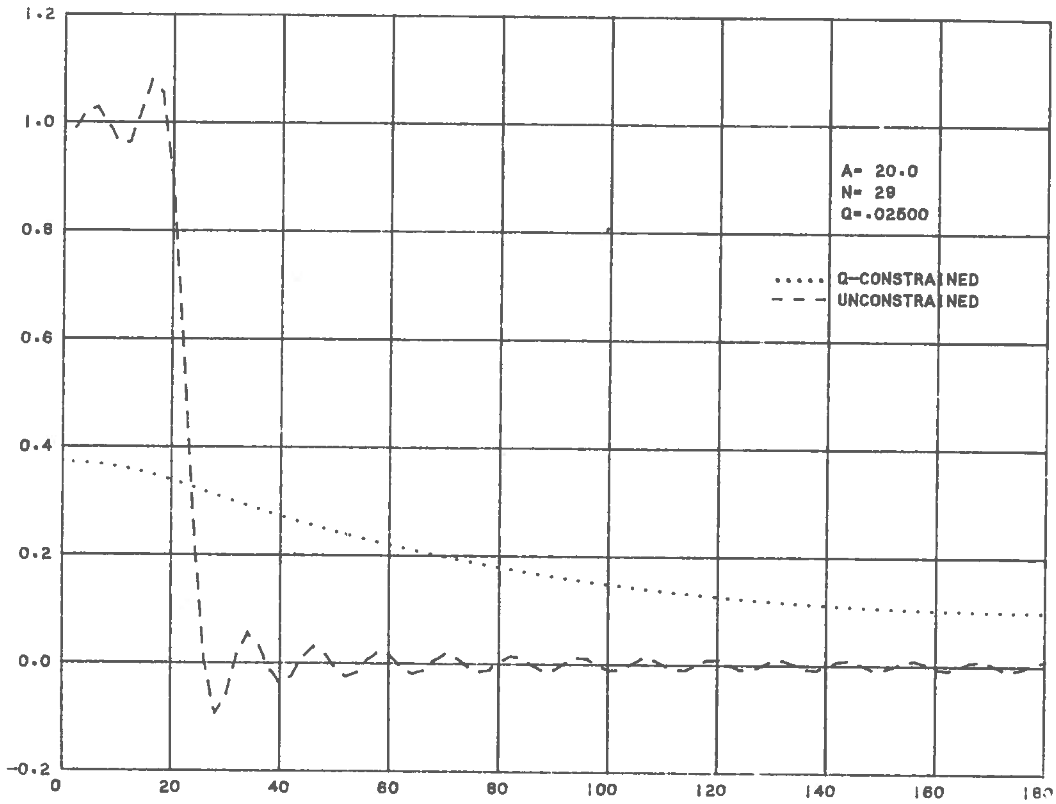


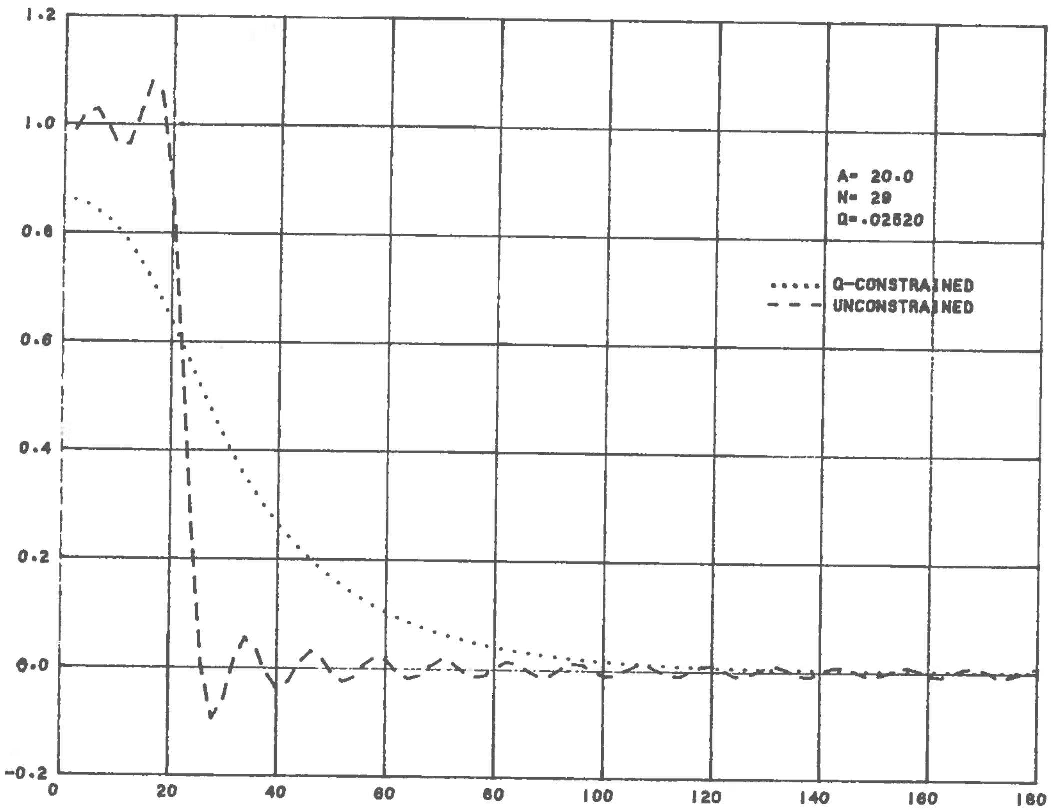
Figs. 12 & 13 Amplitude and phase of the aperture distribution. $Q=.02506$ to $.02515$.



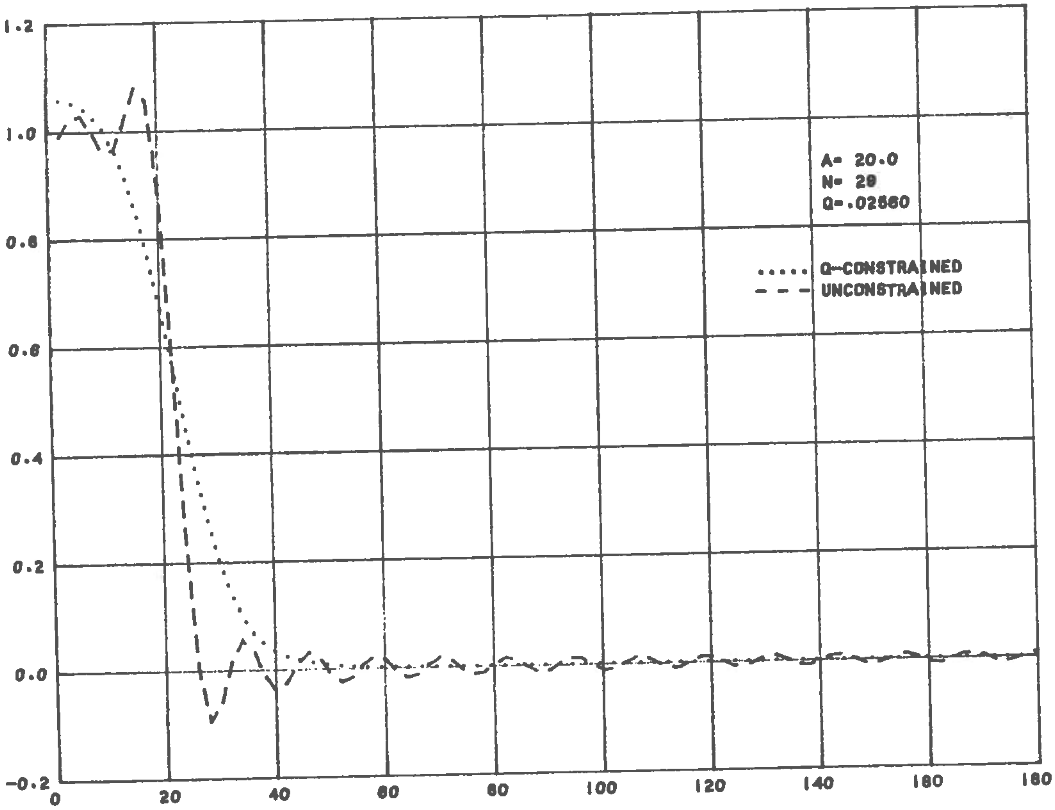


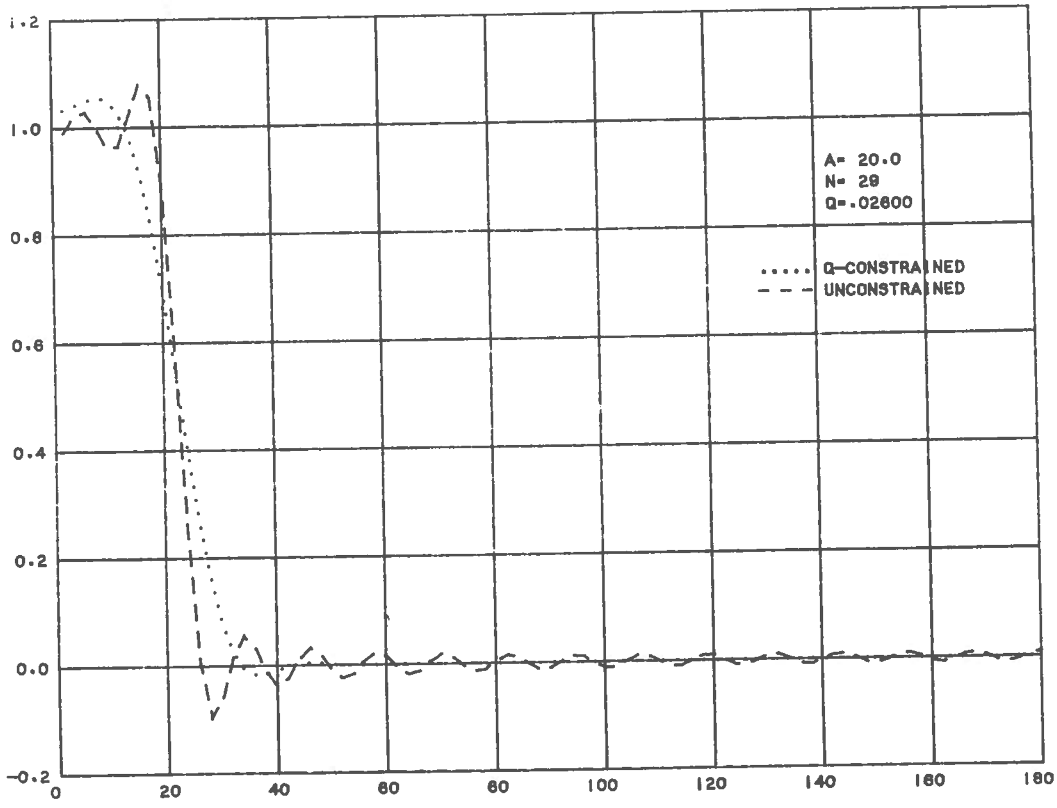
Figs. 14 & 15 Amplitude and phase of the aperture distribution. $Q = .0252$ to $.026$.





Figs. 16 & 17 Realized patterns of a cylindrical aperture; 45° beam; Q = .02500 to .02520.





Figs. 18 & 19 Realized patterns of a cylindrical aperture; 45° beam; Q =.02560 to .02600.

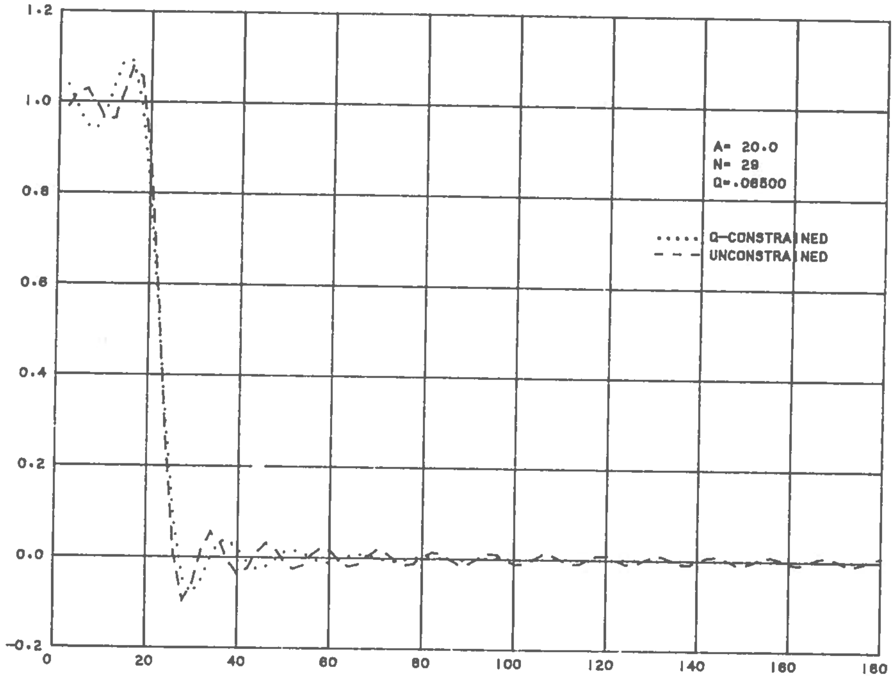


Fig. 20 Realized pattern of a cylindrical aperture; 45° beam; Q = .06500.

IV SYNTHESIS OF A TWO-DIMENSIONAL ELLIPTIC APERTURE

4.1 Analysis for E-polarization

In this case the aperture is an elliptic cylinder described by $u = u_o$ in the elliptical coordinate system $\{u, v, z\}$. See Figure 21. Again we assume that E_z is the only component of electric field. In the region outside the aperture $u = u_o$:

$$E_z(\eta, \xi) = \sum_{0}^{\infty} A_m \, Se_m(c, \eta) \, Re_m^{(3)}(c, \xi) + B_m \, So_m(c, \eta) \, Ro_m^{(3)}(c, \xi) \, . \qquad (4.1;1)$$

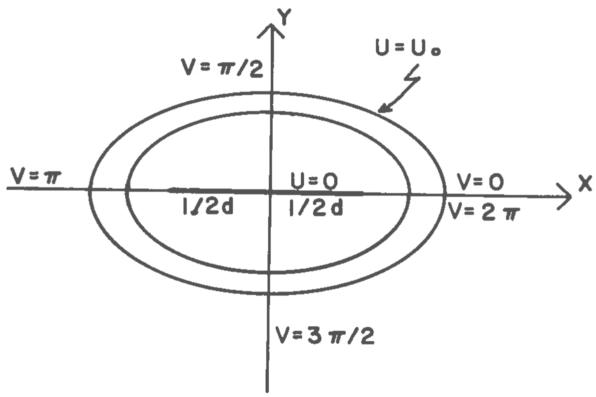


Fig. 21 Elliptic cylindrical coordinates, $u = u_o$ is the aperture surface.

Here we are using Stratton's (1941) definition of the Mathieu functions, and $\eta = \cos\nu$, $\xi = \cosh u$, $c = \frac{1}{2}kd$; d being the interfocal distance.

Since $H_\nu = (i/\omega\mu_o c)\,(\sinh^2 u + \sin^2\nu)^{-\frac{1}{2}}\partial E_z/\partial u$ we have

$$H_\nu = (i/\omega\mu_o c)\sinh u\,(\sinh^2 u + \sin^2\nu)^{-\frac{1}{2}} \sum_o^\infty A_m\, Se_m(c,\eta) Re_m^{(3)'}(c,\xi) + B_m\, So_m(c,\eta) Ro_m^{(3)'}(c,\xi) \quad . \tag{4.1;2}$$

The outward flow of power from the aperture can be computed from

$$S = -\tfrac{1}{2}c\int_o^{2\pi} E_z H_\nu^* (\sinh^2 u_o + \sin^2\nu)^{\frac{1}{2}}\, d\nu$$

with the aid of (1) and (2).

$$S = (i/\omega\mu_o)\sinh u_o \sum_o^\infty |A_m|^2 Re_m^{(3)}(c,\xi_o) Re_m^{(3)'}(c,\xi_o) N_m^e + |B_m|^2 Ro_m^{(3)}(c,\xi_o) Ro_m^{(3)'}(c,\xi_o) N_m^o \, , \tag{4.1;3}$$

where we have employed the orthogonal property of the angle functions:

$$\int_o^{2\pi} \underset{o}{Se}_m(c,\eta)\underset{o}{Se}_n(c,\eta)\, d\nu = N_m^{e,o}\,\delta_{mn} \, . \tag{4.1;4}$$

In (4), N_m^e and N_m^o are normalization constants. On separating into real and imaginary parts and using the Wronskian relation $Re_m^{(2)} Re_m^{(1)'} - Re_m^{(1)} Re_m^{(2)'} = -1/\sinh u$, etc. we get

$$S_r = (1/\omega\mu_o) \sum_o^\infty |A_m|^2 N_m^e + |B_m|^2 N_m^o \tag{4.1;5}$$

$$S_i = (\sinh u_o/\omega\mu_o) \sum_o^\infty |A_m|^2 N_m^e Ze_m + |B_m|^2 N_m^o Zo_m , \tag{4.1;6}$$

with $Ze_m = Re_m^{(1)} Re_m^{(1)'} + Re_m^{(2)} Re_m^{(2)'}$ etc. Again the Q is defined as S_i/S_r.

To find the far field, we employ the asymptotic representations

$\underset{o}{Re}_m^{(3)}(c,\xi) \simeq (c\xi)^{-\frac{1}{2}} .\ e^{i[c\xi-(2m+1)\pi/4]}$ in (1):

$$E_z(\eta,\xi) \simeq (k\xi)^{-\frac{1}{2}} .\ e^{i(c\xi-\pi/4)} \sum_o^\infty [A_m\, Se_m(c,\eta) + B_m\, So_m(c,\eta)] e^{-im\pi/2} \quad . \tag{4.1;7}$$

From (7), the pattern is

$$P(\eta) = \sum_o^\infty a_m \, Se_m(c,\eta) + b_m \, So_m(c,\eta) \tag{4.1;8}$$

with $a_m = e^{-im\pi/2} A_m$, $b_m = e^{-im\pi/2} B_m$. With the desired pattern given by

$$\hat{P}(\eta) = \sum_o^\infty \hat{a}_m \, Se_m(c,\eta) + \hat{b}_m \, So_m(c,\eta) \tag{4.1;9}$$

the process of minimizing the mean square error leads to

$$a_m = \hat{a}_m [\, 1 + \mu \, (Q - \sinh u_o \, Ze_m \,]^{-1} \, , \tag{4.1;10}$$

$$b_m = \hat{b}_m [\, 1 + \mu \, (Q - \sinh u_o \, Zo_m \,]^{-1} \, , \tag{4.1;11}$$

where the Lagrange multiplier μ is to be found from solving the transcendental equation

$$\sum_o^\infty |\hat{a}_m|^2 \cdot |1 + \mu (Q - \sinh u_o \, Ze_m)|^{-2} \cdot (Q - \sinh u_o \, Ze_m) \cdot N_m^e$$
$$+ |\hat{b}_m|^2 \cdot |1 + \mu (Q - \sinh u_o \, Zo_m)|^{-2} \cdot (Q - \sinh u_o \, Zo_m) \cdot N_m^o = 0 \, . \tag{4.1;12}$$

4.2 Discussion

When the desired pattern $\hat{P}(\eta)$ is symmetric about the x axis, the infinite sums of odd functions can be dropped throughout. If $\hat{P}(\eta)$ is symmetric about the y axis, only the Se_m for even m and So_m for odd m need be retained. We also note that $So_o\,(c,\eta) = 0$, so that the odd sums always start at $m = 1$.

Sufficient conditions for minimum error are given by the inequalities

$$1 + \mu (Q - \sinh u_o \, Z{e \atop o}_m) > 0, \; m = 0, \; 1, \ldots\ldots \tag{4.2;1}$$

In contrast with the circular case, it does not seem possible to prove that the $Z{e \atop o}_m$ are monotonic in general. Thus we are not able to establish theoretical bounds on μ, or decide the allowable range of Q in advance. But of course, these questions can be answered by numerical calculation of the $Z{e \atop o}_m$ in any specific case.

4.3 Particular case: infinite slit in conducting plane

This sub-problem of the preceding is of such importance that it calls for a few words of special treatment. For $u_o \to 0$, the elliptical aperture degenerates to a slit of width d; and if

$A_m = 0$ in (4.1;1) the electric field is zero on $\nu = 0$ and $\nu = \pi$. The formulae of Sec. (4.1) can now be used directly, once the quantity $C_m = \lim_{u_o \to 0} (\sinh u_o \cdot Zo_m)$ is evaluated. Since $Ro_m^{(1)}(c, 1) = 0$, and $Ro_m^{(1)'}(c, 1)$ is finite, C_m reduces to $Ro_m^{(2)}(c, 1) \frac{\partial}{\partial u} Ro_m^{(2)}(c, \cosh u)\big|_{u=0}$. These values can be found in a National Bureau of Standards publication (1951).

This problem has also been solved using Mathieu functions by Leonard (1959) for no constraints; and recently with constraint by Rhodes (1972). The latter paper appeared after the present work had been done.

4.4 Analysis of H-polarization

This problem is a trivial variant of the preceding and will not be given here.

V SYNTHESIS OF A PERIODICALLY BLOCKED CIRCULAR APERTURE

5.1 Statement of the problem and analysis

Clearly it is more realistic to have apertures which do not occupy the whole of a co-ordinate surface. In general such apertures lead to problems which do not have an exact solution. But there is one special case which can be solved in closed form; this is an array of similar slots, such as a series of parallel plate waveguides, equally spaced on the circumference of a circular cylinder.

The situation is illustrated in Fig. 22. Here there are N slots, each of angular width $2\psi_o$, whose central axes are $2\pi/N$ radians apart. We assume that $2ka\psi_o \ll 1$; thus E_ϕ across each mouth can be assumed constant*. However, the amplitudes E_s ($s = 0, 1, \ldots. N-1$) are allowed to vary from slot to slot. Consequently the electric field on the cylinder is given by

$$E_\phi(a, \phi) = E_s, \quad \frac{2\pi s}{N} - \psi_o \leqslant \phi \leqslant \frac{2\pi s}{N} + \psi_o \qquad (5.1;1)$$

$$= 0 \ , \ \text{elsewhere.}$$

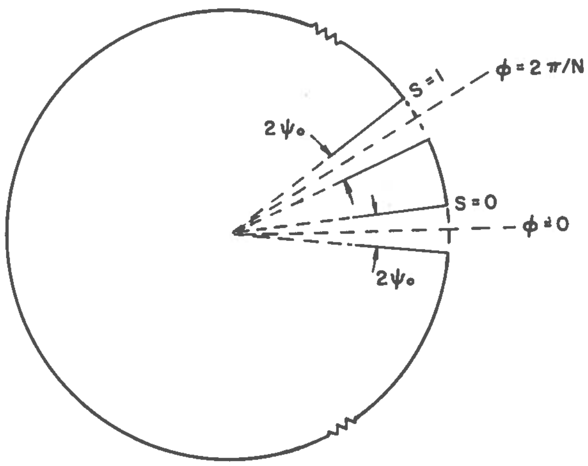


Fig. 22 Slotted cylindrical aperture

* A more realistic field would have a singularity at each edge. The difference in pattern is small for narrow slots.

We also have the representation of (3.5;2):

$$E_\phi(a,\phi) = (-ik/\omega\epsilon_o) \sum_{-\infty}^{\infty} A_m e^{im\phi} H_m^{(1)\prime}(ka) \,, \tag{5.1;2}$$

so after inverting (2) and using (1) we find

$$\frac{-ik}{\omega\epsilon_o} \frac{m\pi}{\sin m\psi_o} A_m H_m^{(1)\prime}(ka) = \sum_{s=0}^{N-1} E_s e^{-2\pi i m s/N} \tag{5.1;3}$$

where $\sin m\psi_o/m$ is to be interpreted as ψ_o if $m = 0$.

As in Sec. (3.5) we have

$$P(\phi) = \sum_{-\infty}^{\infty} a_m e^{im\phi} \,, \tag{5.1;4}$$

$$\hat{P}(\phi) = \sum_{-\infty}^{\infty} \hat{a}_m e^{im\phi} \,, \tag{5.1;5}$$

with $a_m = A_m{}^{-im\pi/2}$. Also

$$S_r = (2/\omega\epsilon_o) \sum_{-\infty}^{\infty} |a_m|^2 \,, \tag{5.1;6}$$

$$S_i = (-\pi ak/\omega\epsilon_o) \sum_{-\infty}^{\infty} |a_m|^2 Z_m \,, \tag{5.1;7}$$

where $Z_m = J_m(ka) J_m{}'(ka) + Y_m(ka) Y_m{}'(ka)$. Thus the quantity to be minimized is

$$\epsilon = 2\pi \sum_{-\infty}^{\infty} |a_m - \hat{a}_m|^2 + \mu \sum_{-\infty}^{\infty} (Q + \tfrac{1}{2}\pi ka Z_m) |a_m|^2 \,. \tag{5.1;8}$$

We now use (3) in (8) and minimize ϵ by varying the quantities E_s. This leads to

$$\sum_{-\infty}^{\infty} \Big\{ [2\pi + \mu(Q + \tfrac{1}{2}\pi ka Z_m)] \cdot |G_m|^{-2} \cdot e^{2\pi i m s/N} \cdot \sum_{s'=0}^{N-1} E_{s'} e^{-2\pi i s' m/N} - \hat{a}_m e^{2\pi i m s/N} / G_m^* \Big\} = 0 \,,$$

$$s = 0, 1, \ldots, N-1 \,. \tag{5.1;9}$$

wherein

$$G_m = \frac{-ik}{\omega\epsilon_o} \cdot \frac{m\pi}{\sin m\psi_o} \cdot e^{im\pi/2} \cdot H_m^{(1)\prime}(ka) \,. \tag{5.1;10}$$

If the sums are re-ordered, (9) can be rewritten as

$$\sum_{s'=0}^{N-1} E_{s'}\, X(s-s') \;=\; Y(s) \qquad (5.1;11)$$

in which

$$X(s) \;=\; \sum_{-\infty}^{\infty} \left[2\pi + \mu\left(Q + \tfrac{1}{2}\pi ka Z_m\right)\right] \cdot |G_m|^{-2} \cdot e^{2\pi i m s/N} \,, \qquad (5.1;12)$$

$$Y(s) \;=\; \sum_{-\infty}^{\infty} \hat{a}_m\, e^{2\pi i m s/N} \,/\, G_m^{*} \;. \qquad (5.1;13)$$

To treat a system such as (11), some of the ideas of Borgiotti and Balzano (1970) can be used. We first note that $X(s+N) = X(s)$ and $Y(s+N) = Y(s)$. It follows that quantities X_t, Y_t can be found such that

$$X(s) \;=\; \sum_{t=0}^{N-1} X_t\, e^{2\pi i s t/N} \,, \qquad (5.1;14)$$

$$Y(s) \;=\; \sum_{t=0}^{N-1} Y_t\, e^{2\pi i s t/N} \;. \qquad (5.1;15)$$

In fact, it is easy to show that

$$\begin{aligned} X_t \;&=\; \sum_{n=-\infty}^{\infty} \left[2\pi + \mu\left(Q + \tfrac{1}{2}\pi ka\, Z_{nN+t}\right)\right] \cdot |G_{nN+t}|^{-2} \,, \\ &=\; U_t + \mu\, V_t \,, \text{ say} \end{aligned} \qquad (5.1;16)$$

and

$$Y_t \;=\; \sum_{n=-\infty}^{\infty} \hat{a}_{nN+t} \,/\, G_{nN+t}^{*} \,. \qquad (5.1;17)$$

If we substitute (14) and (15) in (11) and re-arrange the sums, we get

$$\sum_{t=0}^{N-1} X_t\, e^{2\pi i s t/N} \sum_{s'=0}^{N-1} E_{s'}\, e^{-2\pi i s' t/N} \;=\; \sum_{t=0}^{N-1} Y_t\, e^{2\pi i s t/N} \;. \qquad (5.1;18)$$

A comparison of the coefficients of $e^{2\pi i s t/N}$ in (18) then yields

$$\sum_{s'=0}^{N-1} E_{s'}\, e^{-2\pi i s' t/N} \;=\; Y_t / X_t \,. \qquad (5.1;19)$$

The justification for the last step is that the functions $e^{2\pi ist/N}$ form a complete orthogonal set. Thus if

$$f_t = \sum_{s=0}^{N-1} g_s\, e^{2\pi ist/N}$$

then

$$g_s = N^{-1} \sum_{t=0}^{N-1} f_t\, e^{-2\pi ist/N} ,$$

as can be verified by substitution. These relations allow us to solve (19) for the E_s:

$$E_s = N^{-1} \sum_{t=0}^{N-1} (Y_t/X_t)\, e^{2\pi ist/N} . \qquad (5.1;20)$$

To calculate μ, we first observe that (3) and (19) give $a_m\, G_m = Y_m/X_m$; thus the Q relation

$$\sum_{-\infty}^{\infty} [Q + \tfrac{1}{2}\pi ka\, Z_m] \cdot |a_m|^2 = 0$$

yields, after some algebra,

$$\sum_{s=0}^{N-1} \left|\frac{Y_s}{X_s}\right|^2 V_s = 0 , \qquad (5.1;21)$$

which, with (16) and (17) gives a transcendental equation to be solved for μ.

The realized pattern is then given by

$$P(\phi) = \sum_{-\infty}^{\infty} Y_m (X_m\, G_m)^{-1}\, e^{im\phi} . \qquad (5.1;22)$$

It is not quite straightforward to deduce sufficient conditions for a minimum in this case; for the Hessian matrix is not diagonal when computed with respect to the E_s. However, one can equally well minimize with respect to the quantities Y_t/X_t of (19), and in this case the second derivative condition is just

$$X_t > 0 \quad , t = 0, 1, \ldots , N-1 . \qquad (5.1;23)$$

It would seem difficult to say anything about the behaviour of U_t and V_t as functions of t, in general. But this can always be found by machine for a particular example.

VI SCALAR SYNTHESIS IN THREE DIMENSIONS

Hitherto we have treated only 2 dimensional synthesis problems. But there is reason to expect our methods to be successful in 3 dimensions, providing the corresponding diffraction problem can be solved. Unfortunately not many vector problems are tractable– the sphere and the disc are about the only ones–so we shall have to confine ourselves to scalar problems for the most part.

6.1 Analysis of the prolate spheroid

The aperture comprises the surface $\xi = \xi_0$ in the prolate spheroidal system $\{ \xi, \eta, \phi \}$ depicted in Fig. 23. Here the z axis is the axis of symmetry and the angle ϕ is measured from the x–z plane.

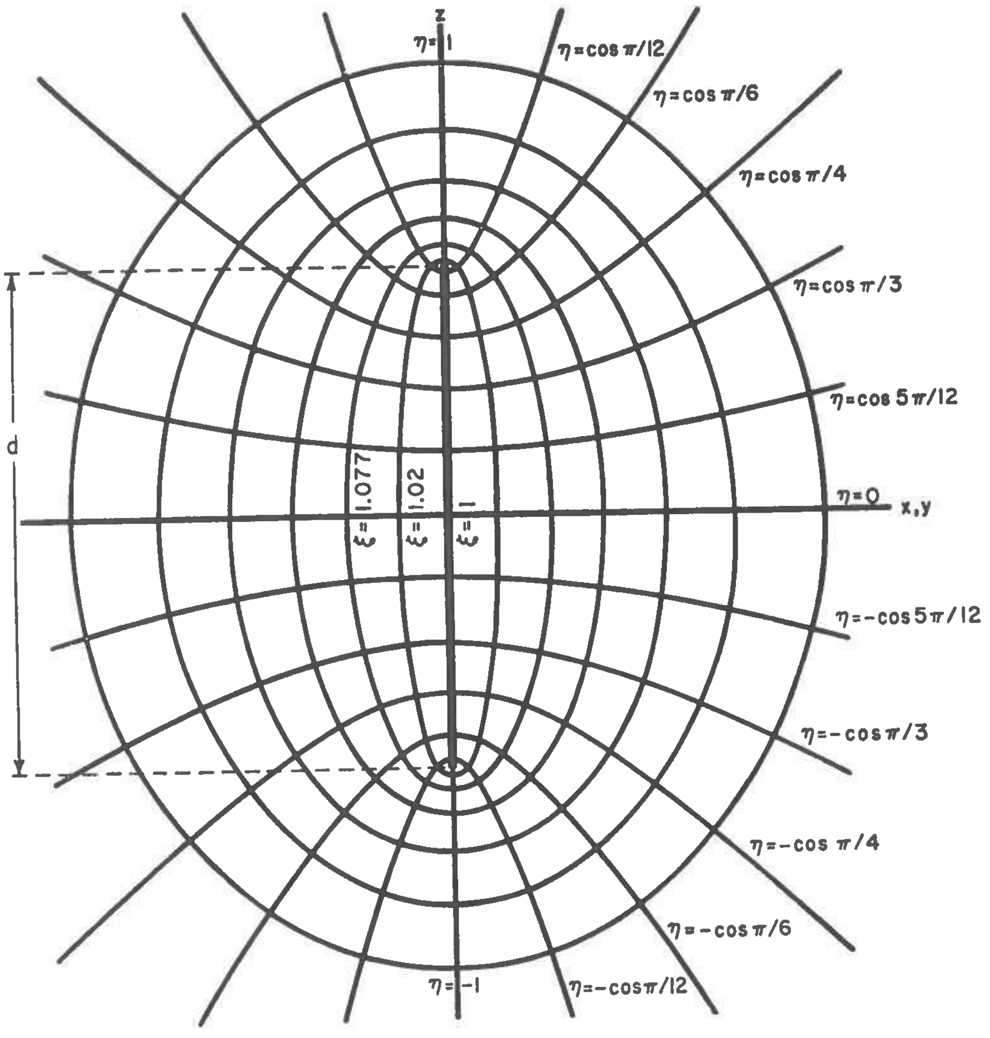


Fig. 23 The prolate spheroidal coordinate system

In the notation of Flammer (1957), a general outgoing wave has the expansion

$$u(\xi,\eta,\phi) = \sum_{m=0}^{\infty} \sum_{n=0}^{m} R_{mn}^{(3)}(c,\xi) S_{mn}(c,\eta) [A_{mn} \cos m\phi + B_{mn} \sin m\phi] \qquad (6.1;1)$$

where $c = \frac{1}{2}kd$ and d is the interfocal distance.

The function $R_{mn}^{(3)}(c,\xi)$ possesses the asymptotic expansion

$$R_{mn}^{(3)}(c,\xi) \simeq (c\xi)^{-1} \exp[ic\xi - \tfrac{1}{2}i\pi(n+1)] \qquad (6.1;2)$$

as $c\xi \to \infty$. Thus the far field expansion of (1) is

$$u(\xi,\eta,\phi) = \frac{e^{ic\xi}}{c\xi} \sum_{m=0}^{\infty} \sum_{n=0}^{m} e^{-\frac{1}{2}i\pi(n+1)} S_{mn}(c,\eta)[A_{mn} \cos m\phi + B_{mn} \sin m\phi] \qquad (6.1;3)$$

Since $c\xi \to r$ (the radial distance), the pattern function becomes

$$P(\eta,\phi) = \sum_{m=0}^{\infty} \sum_{n=0}^{m} S_{mn}(c,\eta)[a_{mn} \cos m\phi + b_{mn} \sin m\phi], \qquad (6.1;4)$$

where we write $a_{mn} = e^{-\frac{1}{2}i(n+1)} A_{mn}$ etc. Again we let $\hat{P}(\eta,\phi)$ be the wanted pattern with constants $\hat{a}_{mn}$, $\hat{b}_{mn}$ found from

$$\begin{matrix}\hat{a}_{mn} \\ \hat{b}_{mn}\end{matrix} = (\epsilon_m / 2\pi N_{mn}) \int_o^{2\pi} \begin{matrix}\cos m\phi \\ \sin m\phi\end{matrix} \, d\phi \int_{-1}^{1} S_{mn}(c,\eta) \hat{P}(\eta,\phi) \, d\eta \qquad (6.1;5)$$

Here $\epsilon_m = 1$ for $m = 0$ and is 2 for $m > 0$. N_{mn} is the normalization factor of the $S_{mn}(c,\eta)$, viz:

$$\int_{-1}^{1} S_{mn}(c,\eta) S_{mn'}(c,\eta) \, d\eta = \delta_{nn'} N_{mn} .$$

The power in the aperture is proportional to $iu \, \partial u^*/\partial n$ integrated over the surface of the spheroid. We have $\frac{\partial}{\partial n} = \frac{1}{c}\sqrt{\frac{\xi^2-1}{\xi^2-\eta^2}} \frac{\partial}{\partial \xi}$, while the element of area is $da = c^2 \sqrt{(\xi^2-\eta^2)(\xi^2-1)} \, d\eta d\phi$.

Then

$$S = 2\pi i c (\xi_o^2 - 1) \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{R_{mn}^{(3)}(c,\xi_o) R_{mn}^{(4)'}(c,\xi_o) N_{mn}}{\epsilon_m} \Big\{ |a_{mn}|^2 + |b_{mn}|^2 (1-\delta_{m0}) \Big\} . \qquad (6.1;6)$$

In view of the Wronskian $R_{mn}^{(1)} R_{mn}^{(2)'} - R_{mn}^{(2)} R_{mn}^{(1)'} = c^{-1} (\xi^2 - 1)^{-1}$, we have

$$S_r = 2\pi \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{N_{mn}}{\epsilon_m} \left\{ |a_{mn}|^2 + |b_{mn}|^2 (1 - \delta_{m0}) \right\}, \tag{6.1;7}$$

$$S_i = 2\pi c (\xi_o^2 - 1) \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{Z_{mn} N_{mn}}{\epsilon_m} \left\{ |a_{mn}|^2 + |b_{mn}|^2 (1 - \delta_{m0}) \right\}, \tag{6.1;8}$$

where $Z_{mn} = R_{mn}^{(1)} R_{mn}^{(1)'} + R_{mn}^{(2)} R_{mn}^{(2)'}$. With $Q = S_i/S_r$, the minimization process leads to

$$a_{mn} = \hat{a}_{mn} \left\{ 1 + \mu[Q - c(\xi_o^2 - 1) Z_{mn}] \right\}^{-1}, \tag{6.1;9}$$

$$b_{mn} = \hat{b}_{mn} \left\{ 1 + \mu [Q - c(\xi_o^2 - 1) Z_{mn}] \right\}^{-1}. \tag{6.1;10}$$

As usual μ is found by substituting (9) and (10) in the expression for Q, while the conditions for minimum error are

$$1 + \mu[Q - c(\xi_o^2 - 1) Z_{mn}] > 0 . \tag{6.1;11}$$

Again it is not possible to say much about bounds in the general case.

6.2 Special case: the sphere

A case of some interest is the sphere, the solution for which is got by letting $c \to 0$, $c\xi \to r$, $c\xi_o \to a$ in the preceding. We have $S_{mn}(c,\eta) \to P_n^m(\eta)$, $R_{mn}^{(3,4)}(c,\xi) \to h_n^{(1,2)}(kr)$, $N_{mn} = \frac{2}{2n+1} \cdot \frac{(n+m)!}{(n+m)!}$; where $P_n^m(\eta)$ is the associated Legendre polynomial and $h_n^{(1,2)}(kr)$ the spherical Hankel functions as defined in Stratton (1941), pages 401 and 404 respectively.

It would also be interesting to recover the line 'aperture' considered in Sec. (2.1) from the foregoing, thus getting a slightly different version of Rhodes' problem. But unfortunately, the situation does not remain valid in the limit $\xi_o \to 1$ owing to S_i becoming infinite.

6.3 Analysis of the oblate spheroid

The coordinate system $\{\xi, \eta, \phi\}$ is shown in Fig. 24, where z is the symmetry axis and ϕ is measured from the x-z plane.

The solution can be written down directly from that of Sec. 6.1 by making the changes $c \to -ic$, $\xi \to i\xi$ throughout. In so doing, it should be noted that the primed quantities in Sec. 6.1 are derivatives with respect to ξ.

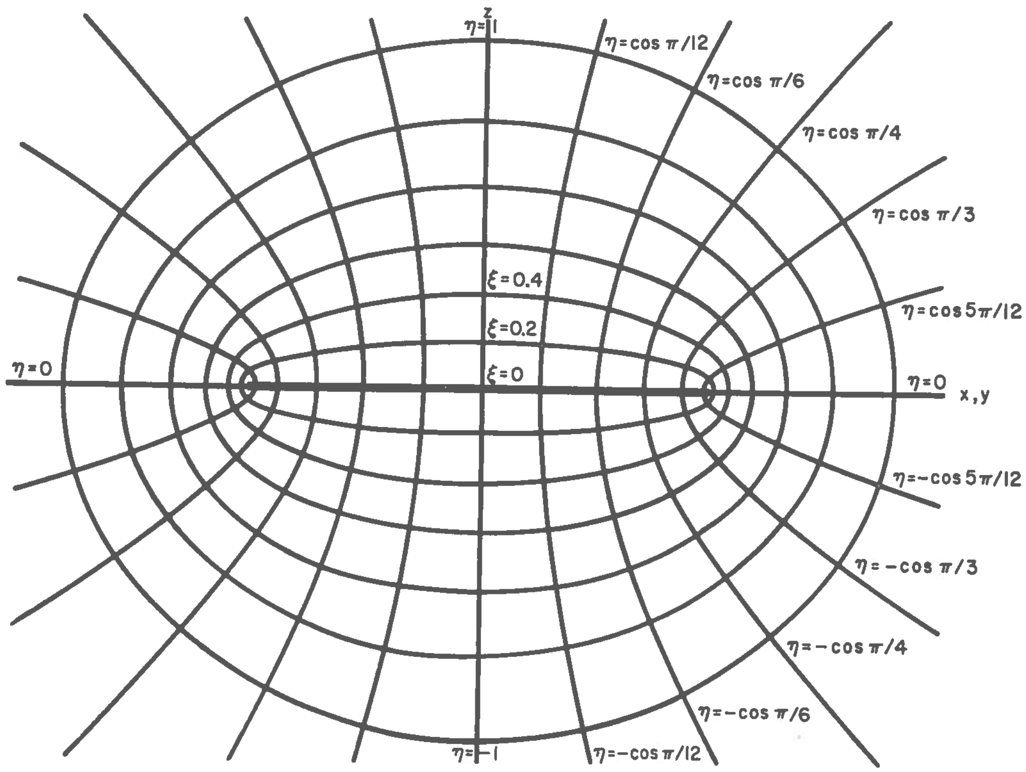


Fig. 24 The oblate spheroidal coordinate system

6.4 Special case: the circular aperture

The important case of a circular aperture of diameter d in a plane can be obtained by letting $\xi_o \to 0$ in the preceding. The different boundary conditions on the plane $u = 0$ or $\partial u/\partial n = 0$ can be handled by restricting the sums to $n-m$ odd and even respectively. In this case Z_{mn} remains finite, becoming just

$$Z_{mn} \;=\; -R^{(2)}_{mn}(-ic, io)\, R^{(2)'}_{mn}(-ic, io)\ . \tag{6.4;1}$$

These values are given by Flammer (1957), page 42, where it should be noted that Flammer's prime denotes derivative with respect to ξ.

While the vector diffraction problem of a circular aperture has been solved rigorously, it does not appear possible to handle the synthesis problem with our methods. This is because the vector oblate functions do not have enough orthogonality properties to allow the aperture power to be calculated simply. But we might note that Fante (1970) solved the problem using generalized prolate spheroidal functions under a somewhat weaker constraint.

VII THE VECTOR PROBLEM FOR THE SPHERE*

7.1 Analysis of the problem

This is the only three dimensional vector problem that appears solvable. In the spherical coordinate system $\{r, \theta, \phi\}$ the aperture is defined by $r = a$. In the region $r \geqslant a$ the electric field will be given by

$$\mathbf{E} = \sum_{\ell=0}^{\infty} \sum_{k=0}^{\ell} [A_{k\ell}\, \mathbf{m}^{(3)}_{ok\ell} + B_{k\ell} \mathbf{m}^{(3)}_{ek\ell} + C_{k\ell}\, \mathbf{n}^{(3)}_{ok\ell} + D_{k\ell}\, \mathbf{n}^{(3)}_{ek\ell}], \qquad (7.1;1)$$

where $\mathbf{m}$, $\mathbf{n}$ are vector spherical wave functions as defined by Stratton (1941), page 416. $A_{k\ell}$, $B_{k\ell}$, $C_{k\ell}$, $D_{k\ell}$ are undetermined coefficients. From (1) the magnetic field is

$$\mathbf{H} = (-ik/\omega\mu_o) \sum_{\ell=0}^{\infty} \sum_{k=0}^{\ell} [A_{k\ell}\mathbf{n}^{(3)}_{ok\ell} + B_{k\ell}\, \mathbf{n}^{(3)}_{ek\ell} + C_{k\ell}\, \mathbf{m}^{(3)}_{ok\ell} + D_{k\ell}\, \mathbf{m}^{(3)}_{ek\ell}]\,. \qquad (7.1;2)$$

As $r \to \infty$ we have

$$\mathbf{m}^{(3)}_{\substack{e \\ o}k\ell} \sim \left\{ \mp \frac{k}{\sin\theta} P^k_\ell (\cos\theta) \begin{matrix}\sin \\ \cos\end{matrix}\, k\phi\, \hat{\theta} - \frac{\partial P^k_\ell}{\partial\theta} (\cos\theta) \begin{matrix}\cos \\ \sin\end{matrix}\, k\phi\, \hat{\phi} \right\} (-i)^{\ell+1}\, e^{ikr}/kr$$

$$= (-i)^{\ell+1}\, \mathbf{M}_{\substack{e \\ o}k\ell}\, e^{ikr}/kr, \text{ say, and} \qquad (7.1;3)$$

$$\mathbf{n}^{(3)}_{\substack{e \\ o}k\ell} \sim \left\{ \frac{\partial P^k_\ell}{\partial\theta} (\cos\theta) \cdot \begin{matrix}\cos \\ \sin\end{matrix}\, k\phi\, \hat{\theta} \mp \frac{k}{\sin\theta} P^k_\ell (\cos\theta) \begin{matrix}\sin \\ \cos\end{matrix}\, k\phi\, \hat{\phi} \right\} (-i)^{\ell} \frac{e^{ikr}}{kr}$$

$$= (-i)^{\ell}\, \mathbf{N}_{\substack{e \\ o}k\ell}\, e^{ikr}/kr. \qquad (7.1;4)$$

where the upper rows of symbols are for the even functions. It follows that the far field is

$$\mathbf{P}(\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{k=0}^{\ell} [a_{k\ell}\, \mathbf{M}_{ok\ell} + b_{k\ell}\, \mathbf{M}_{ek\ell} + c_{k\ell}\, \mathbf{N}_{ok\ell} + d_{k\ell}\, \mathbf{N}_{ek\ell}] \qquad (7.1;5)$$

where $a_{k\ell} = (-i)^{\ell+1} A_{k\ell}$, $b_{k\ell} = (-i)^{\ell+1} B_{k\ell}$, $c_{k\ell} = (-i)^{\ell} C_{k\ell}$, $d_{k\ell} = (-i)^{\ell} D_{k\ell}$.

The desired pattern is

$$\hat{\mathbf{P}}(\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{k=0}^{\ell} [\hat{a}_{k\ell}\, \mathbf{M}_{ok\ell} + \hat{b}_{k\ell}\, \mathbf{M}_{ek\ell} + \hat{c}_{k\ell}\, \mathbf{N}_{ok\ell} + \hat{d}_{k\ell}\, \mathbf{N}_{ek\ell}]\,. \qquad (7.1;6)$$

---

* It is a pleasure to record the contributions of T.A. Gough to this section: he checked the mathematics, found mistakes and did the programming.

To find the coefficients $\hat{a}_{k\ell}$, $\hat{b}_{k\ell}$, $\hat{c}_{k\ell}$, $\hat{d}_{k\ell}$ for a given $\hat{\mathbf{P}}(\theta,\phi)$, one applies the orthogonality relations

$$\int_0^{2\pi} d\phi \int_0^{\pi} \mathbf{M}_{\substack{e\\o}k\ell} \cdot \mathbf{M}_{\substack{e\\o}k'\ell'} \sin\theta \, d\theta = \delta_{kk'}\,\delta_{\ell\ell'}\, X^{e,o}_{k\ell} ,$$

$$\int_0^{2\pi} d\phi \int_0^{\pi} \mathbf{N}_{\substack{e\\o}k\ell} \cdot \mathbf{N}_{\substack{e\\o}k'\ell'} \sin\theta \, d\theta = \delta_{kk'}\,\delta_{\ell\ell'}\, X^{e,o}_{k\ell} ,$$

where

$$X^{e}_{k\ell} = X^{o}_{k\ell} = \frac{2\pi\ell(\ell+1)}{2\ell+1}\;\frac{(\ell+k)!}{(\ell-k)!} , \quad k > 0$$

$$X^{e}_{0\ell} = \frac{4\pi\ell(\ell+1)}{2\ell+1} ,$$

$$X^{o}_{0\ell} = 0$$

Any product of even and odd functions, or **M** and **N** functions integrates to zero. Then the coefficients are

$$\hat{a}_{k\ell} = \frac{1}{X^{o}_{k\ell}} \int_0^{2\pi} d\phi \int_0^{\pi} \mathbf{M}_{ok\ell} \cdot \hat{\mathbf{P}}(\theta,\phi) \sin\theta \, d\theta \qquad (7.1;7)$$

and so forth.

To find the power flow outward through the aperture, we have to calculate

$$S = \tfrac{1}{2}a^2 \int_0^{2\pi} d\phi \int_0^{\pi} \mathbf{E} \times \mathbf{H}^* \cdot \hat{r} \, \sin\theta \, d\theta \, . \qquad (7.1;8)$$

We begin by noting that no radial components of **E** or **H** contribute to the integral. We can therefore rewrite $\mathbf{m}_{\substack{e\\o}}$ and $\mathbf{n}_{\substack{e\\o}}$ for integration purposes as

$$\mathbf{m}^{(3)}_{\substack{e\\o}k\ell} = \mathbf{M}_{\substack{e\\o}k\ell} \; h^{(1)}_{\ell}(ka) \qquad (7.1;9)$$

$$\begin{aligned}\mathbf{n}^{(3)}_{\substack{e\\o}k\ell} &= \mathbf{N}_{\substack{e\\o}k\ell} \cdot \frac{1}{ka} \cdot \frac{\partial}{\partial r}\,[r\,h^{(1)}_{\ell}(kr)]_{r=a} \, . \\ &= \mathbf{N}_{\substack{e\\o}k\ell}\, f_{\ell} \, , \quad \text{say} \, .\end{aligned} \qquad (7.1;10)$$

Now $\mathbf{E} \times \mathbf{H} \cdot \hat{r} = \mathbf{E} \cdot \mathbf{H} \times \hat{r}$. Since $\mathbf{M} \times \hat{r} = -\mathbf{N}$ and $\mathbf{N} \times \hat{r} = \mathbf{M}$, we have

$$\mathbf{H} \times \hat{r} = \frac{-ik}{\omega\mu_o} \sum_{\ell=0}^{\infty} \sum_{k=0}^{\ell} [A_{k\ell}\,\mathbf{M}_{ok\ell} f^{(1)}_{\ell} + B_{k\ell}\,\mathbf{M}_{ek\ell} f^{(1)}_{\ell} - C_{k\ell}\,\mathbf{N}_{ok\ell}\, h^{(1)}_{\ell} - D_{k\ell}\,\mathbf{N}_{ek\ell}\, h^{(1)}_{\ell}]$$

$$(7.1;11)$$

$$\mathbf{E} = \sum_{\ell=0}^{\infty} \sum_{k=0}^{\ell} [A_{k\ell} \mathbf{M}_{ok\ell} h_\ell^{(1)} + B_{k\ell} \mathbf{M}_{ek\ell} h_\ell^{(1)} + C_{k\ell} \mathbf{N}_{ok\ell} f_\ell^{(1)} + D_{k\ell} \mathbf{N}_{ek\ell} f_\ell^{(1)}] \quad . \qquad (7.1;12)$$

When (11) and (12) are substituted in (8) we obtain

$$S = \frac{ika^2}{2\omega\mu_o} \sum_{\ell=0}^{\infty} \sum_{k=0}^{\ell} [\,|a_{k\ell}|^2 X_{k\ell}^o + |b_{k\ell}|^2 X_{k\ell}^e\,] h_\ell^{(1)} f_\ell^{(2)}$$

$$- [\,|c_{k\ell}|^2 X_{k\ell}^o + |d_{k\ell}|^2 X_{k\ell}^e\,] h_\ell^{(2)} f_\ell^{(1)}. \qquad (7.1;13)$$

Since $f_\ell^{(1)}(ka) = h_\ell^{(1)'}(ka) + (ka)^{-1} h_\ell^{(1)}(ka)$, we have

$$S_r = \frac{1}{2\omega\mu_o k} \sum_{\ell=0}^{\infty} \sum_{k=0}^{\ell} \left\{ [\,|a_{k\ell}|^2 + |c_{k\ell}|^2] X_{k\ell}^o + [|b_{k\ell}|^2 + |d_{k\ell}|^2] X_{k\ell}^e \right\}, \qquad (7.1;14)$$

$$S_i = \frac{ka^2}{2\omega\mu_o} \sum_{\ell=0}^{\infty} \sum_{k=0}^{\ell} \left\{ [\,|a_{k\ell}|^2 - |c_{k\ell}|^2] X_{k\ell}^o + [|b_{k\ell}|^2 - |d_{k\ell}|^2] X_{k\ell}^e \right\} Z_\ell \qquad (7.1;15)$$

where

$$Z_\ell = \tfrac{1}{2} [j_\ell^2(ka) + y_\ell^2(ka)]' + (ka)^{-1} [j_\ell^2(ka) + y_\ell^2(ka)] \quad . \qquad (7.1;16)$$

We would ordinarily set $Q = S_i/S_r$. But S_i now contains both positive and negative terms, thus it would be possible to reduce it to zero by letting $|a_{k\ell}|^2 = |b_{k\ell}|^2$, $|c_{k\ell}|^2 = |d_{k\ell}|^2$. Since this can be done in an infinity of ways, it would appear that this Q restriction is fairly meaningless. However, an expression closely analogous to our previous Q's would be obtained if the negative signs of (15) were reversed. This we arbitrarily do.

The expression for the mean square error is

$$\epsilon = \sum_{\ell=0}^{\infty} \sum_{k=0}^{\ell} \left\{ |a_{k\ell} - \hat{a}_{k\ell}|^2 X_{k\ell}^o + |b_{k\ell} - \hat{b}_{k\ell}|^2 X_{k\ell}^e \right.$$

$$\left. + |c_{k\ell} - \hat{c}_{k\ell}|^2 X_{k\ell}^o + |d_{k\ell} - \hat{d}_{k\ell}|^2 X_{k\ell}^e \right\} \quad . \qquad (7.1;17)$$

Minimizing in the usual manner, we are led to

$$a_{k\ell} = \hat{a}_{k\ell} [1 + \mu(Q - k^2 a^2 Z_\ell)]^{-1} \text{ etc.} \qquad (7.1;18)$$

while μ is found from

$$\sum_{\ell=0}^{\infty} \sum_{k=0}^{\ell} \frac{(Q - k^2 a^2 Z_\ell)}{[1 + \mu(Q - k^2 a^2 Z_\ell)]^2} \cdot \left\{ |\hat{a}_{k\ell}|^2 X_{k\ell}^o + |\hat{b}_{k\ell}|^2 X_{k\ell}^e + |\hat{c}_{k\ell}|^2 X_{k\ell}^o + |\hat{d}_{k\ell}|^2 X_{k\ell}^e \right\} = 0$$

$$(7.1;19)$$

7.2 Subsidiary proofs and bounds

In the spherical system we are once again able to supply accurate bounds for the various quantities. We begin by investigating the behaviour with ℓ of

$$Z_\ell = \tfrac{1}{2}[j_\ell^2(ka) + y_\ell^2(ka)]' + (1/ka)[j_\ell^2(ka) + y_\ell^2(ka)] \quad .$$

Similarly to (3.2;1) we obtain the integral representation

$$Z_\ell = \frac{4}{\pi^2 k^2 a^2} \int_0^\infty \cosh(2\ell+1)t\,[K_o(2ka\sinh t) - 2ka\sinh t \cdot K_1(2ka\sinh t)]\,dt. \quad (7.2;1)$$

For small t the K_o term of the integrand dominates, while the K_1 term becomes larger as $t \to \infty$. Consequently for $t = \tau$ (independent of ℓ), the two terms are equal. We split the range of integration at τ and write $A(\ell) = \int_0^\tau$, $B(\ell) = \int_\tau^\infty$ so that $\frac{1}{4}\pi^2 k^2 a^2 Z_\ell = A(\ell) + B(\ell)$. $A(\ell)$ is positive and $B(\ell)$ negative. For $\ell = 0$, a short calculation gives $Z_\ell = 0$. Thus $A(0) = -B(0)$. As ℓ increases, the increase in the multiplier $\cosh(2\ell+1)t$ is everywhere greater in $B(\ell)$ than $A(\ell)$. Therefore Z_ℓ is monotonically decreasing and $\leqslant 0$.

It is now clear that Q must be chosen negative. If we assume that only a finite number of harmonics can be generated, i.e., the upper limit of the sum on ℓ is L, then from (7.1;19)

$$k^2 a^2 Z_L < Q < 0 \qquad (7.2;2)$$

becomes the allowable range of Q.

The conditions for a minimum mean square error are

$$1 + \mu[Q - k^2 a^2 Z_\ell] > 0, \quad \ell = 0,\ 1,\ \ldots\ldots\ L \qquad (7.2;3)$$

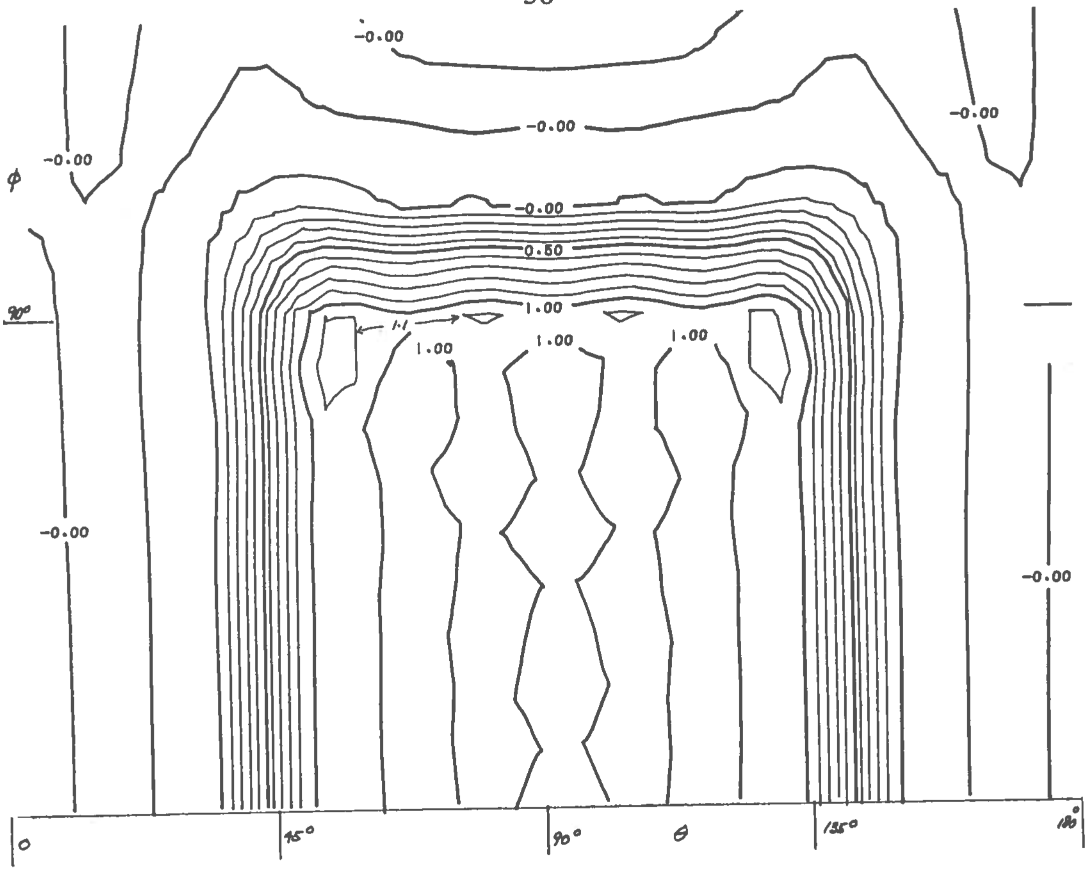
which leads to

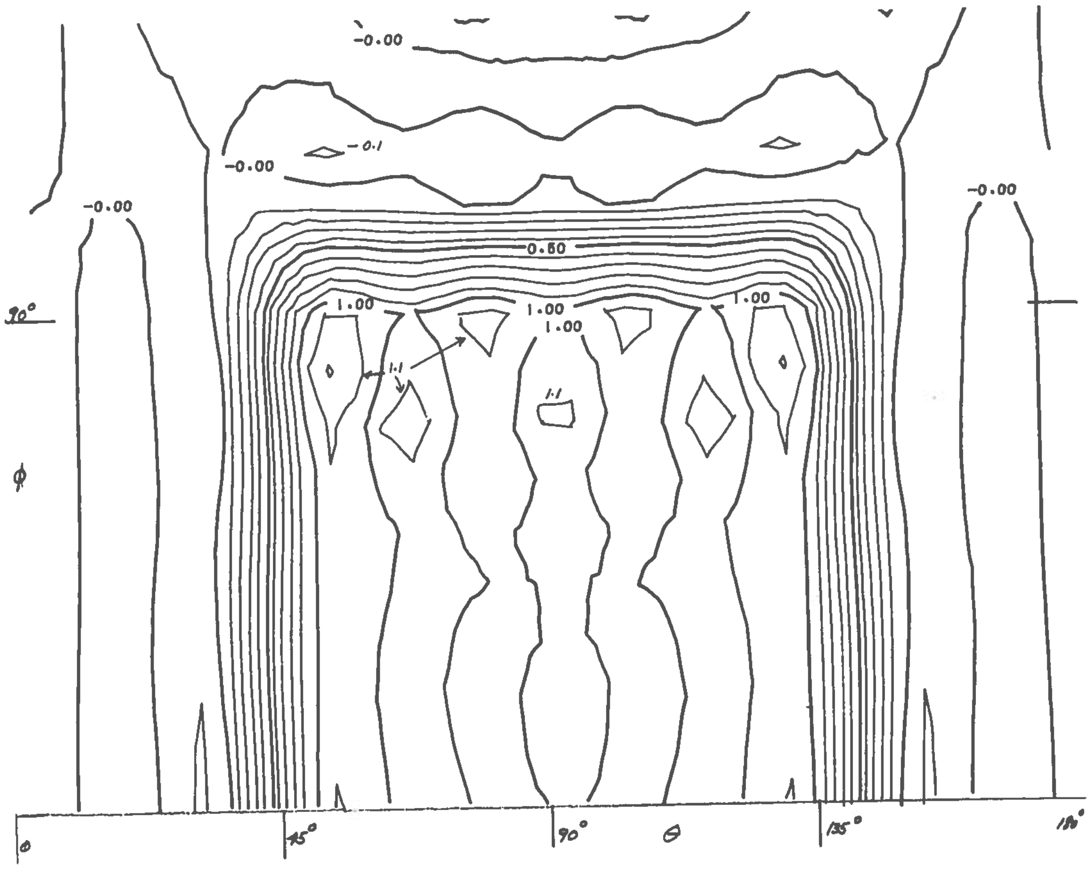
$$-(Q - k^2 a^2 Z_L)^{-1} < \mu < -Q^{-1} \qquad (7.2;4)$$

as the allowable range of μ.

7.3 Numerical results

Figures 25 and 26 show realized two dimensional patterns in the form of contour plots for differing Q values. The desired pattern is unity in $45° \leqslant \theta \leqslant 135°$, $-90° \leqslant \phi \leqslant 90°$ and zero outside. The ka value is 20, and the contour interval is 0.1. In the curtailed region $\phi > 135°$ approximately, only zero contours exist.





Figs. 25 & 26 Two dimensional patterns of a spherical aperture.

VIII SOME PROBLEMS NOT SOLVABLE IN CLOSED FORM

Even when a problem is not solvable exactly, but is left in the implicit form mentioned earlier, it may still be entirely feasible to obtain useful results. Normally this means that a matrix will have to be inverted, whose dimension will be the number of elements in an array or the number of harmonics in an aperture, unless some special symmetries exist. Thus in practice we are limited to reasonably small arrays or apertures. In this section we consider a pair of problems of this type – an arbitrary array of similarly oriented dipoles and unequally spaced similar slots on a metal cylinder. For the latter problem we use our normal Q constraint, but we find Uzsoky's Q more useful for the former.

8.1 A discrete array of infinitesimal dipoles

Here we have N similarly oriented but arbitrarily placed infinitesimal dipoles. Let them occupy positions r_n, θ_n, ϕ_n, in the spherical coordinate system $\{ r, \theta, \phi, \}$ and denote their driving currents by J_n.

The far-field pattern of such an array is well known. It is

$$P(\theta,\phi) = g(\theta,\phi) \sum_{n=1}^{N} J_n \exp[-ikr_n \cos\alpha_n] , \tag{8.1;1}$$

where $g(\theta,\phi)$ is the element pattern ($\sin\theta$ for a z-directed dipole) and

$$\cos\alpha_n = \sin\theta \sin\theta_n \cos(\phi-\phi_n) + \cos\theta \cos\theta_n . \tag{8.1;2}$$

If $\hat{P}(\theta,\phi)$ is the desired pattern, the mean square error turns out to be

$$\epsilon = \sum_{m=1}^{N} \sum_{n=1}^{N} b_{mn} J_m J_n{}^* - \sum_{n=1}^{N} J_n a_n{}^* - \sum_{n=1}^{N} J_n{}^* a_n + |E|^2 , \tag{8.1;3}$$

where

$$b_{mn} = \int_0^{2\pi} d\phi \int_0^{\pi} |g(\theta,\phi)|^2 \exp[ik(r_n \cos\alpha_n - r_m \cos\alpha_m)] \sin\theta \, d\theta, \tag{8.1;4}$$

$$a_n = \int_0^{2\pi} d\phi \int_0^{\pi} \hat{P}(\theta,\phi) g^* (\theta,\phi) \exp[ikr_n \cos\alpha_n] \sin\theta \, d\theta , \tag{8.1;5}$$

$$|E|^2 = \int_0^{2\pi} d\phi \int_0^{\pi} |\hat{P}(\theta,\phi)|^2 \sin\theta \, d\theta . \tag{8.1;6}$$

For the Q constraint we now use

$$Q = \sum_1^N |J_n|^2 \Big/ \sum_{n=1}^{N} \sum_{m=1}^{N} b_{mn} J_m J_n{}^*. \tag{8.1;7}$$

Minimizing in the usual manner, we are led to

$$\sum_{n=1}^{N} [(1+\mu Q)b_{mn} - \mu\delta_{mn}]J_n = a_m \quad , \tag{8.1;8}$$

or in an obvious matrix notation

$$[(1+\mu Q)\mathbf{B} - \mu\mathbf{I}]\,\mathbf{J} = \mathbf{A} \quad . \tag{8.1;9}$$

Thus we have to invert an $N \times N$ matrix to find J. This is complicated in the present instance by the presence of the unknown quantity μ in the matrix. The difficulty can be overcome in the following way. Consider the matrix equation

$$\mathbf{B\,x} = \lambda\mathbf{x} \quad . \tag{8.1;10}$$

In general, it possesses N orthonormal eigenvectors $\mathbf{x}_i$ and N eigenvalues λ_i. Since it is readily shown that $\mathbf{B}$ is symmetric and real, these are real.

Suppose that

$$\mathbf{J} = \sum_{i=1}^{N} c_i\,\mathbf{x}_i \tag{8.1;11}$$

and

$$\mathbf{A} = \sum_{i=1}^{N} d_i\,\mathbf{x}_i \quad ,$$

where the d_i are known and the c_i are to be found. Then

$$[(1+\mu Q)\mathbf{B} - \mu\mathbf{I}]\,\mathbf{J} = (1+\mu Q)\mathbf{B}\sum c_i\mathbf{x}_i - \mu\sum c_i\mathbf{x}_i \tag{8.1;12}$$

which by virtue of (10) is

$$= (1+\mu Q)\sum c_i\lambda_i\mathbf{x}_i - \mu\sum c_i\mathbf{x}_i \,. \tag{8.1;13}$$

On comparing coefficients of $\mathbf{x}_i$ we obtain

$$c_i = d_i[(1+\mu Q)\lambda_i - \mu]^{-1} \,. \tag{8.1;14}$$

In terms of c_i, the Q relation (7) becomes

$$Q = \sum_{i=1}^{N} |c_i|^2 \Big/ \sum_{i=1}^{N} \lambda_i\,|c_i|^2 \,. \tag{8.1;15}$$

If (14) is inserted in (15), the equation for μ becomes

$$\sum_{i=1}^{N} \frac{|d_i|^2 \,(Q\lambda_i - 1)}{[(1+\mu Q)\,\lambda_i - \mu]^2} \;=\; 0\,. \qquad (8.1;16)$$

Again we note that Q must be properly chosen if (16) is to have a solution. Since computer routines exist for finding eigenvalues and eigenvectors of matrices, the problem is essentially solved.

Finally we note that the sufficient conditions for minimum error are:

$$(1+\mu Q) b_{mm} - \mu > 0, \quad m = 1, \quad 2 \,......., N \;. \qquad (8.1;17)$$

8.2 Computation of the quantities b_{mn}

The matrix elements b_{mn} have been computed several times in recent years – by Lo *et al* (1966), Forman (1970), and Hansen (1972). In each case the results are for planar arrays. But the results are easily extended to arbitrary arrays:

$$b_{mm} = 8\pi/3, \qquad (8.2;1)$$

$$b_{mn} = -4\pi \left\{ \frac{2\cos kr_{mn}}{(kr_{mn})^2} - \frac{2\sin kr_{mn}}{(kr_{mn})^3} + \frac{\bar{r}_{mn}{}^2}{r_{mn}{}^2} \left[- \frac{\sin kr_{mn}}{kr_{mn}} - \frac{3\cos kr_{mn}}{(kr_{mn})^2} + \right.\right.$$

$$\left.\left. \frac{3\sin kr_{mn}}{(kr_{mn})^3} \right] \right\}, \; m \neq n\,. \qquad (8.2;2)$$

Here r_{mn} is the inter-element distance while $\bar{r}_{mn}$ is the projection of r_{mn} on the plane $\theta = \pi/2$, viz,

$$\bar{r}_{mn}{}^2 = r_m{}^2 \sin^2\theta_m + r_n{}^2 \sin^2\theta_n - 2 r_m r_n \sin\theta_m \sin\theta_n \cos(\phi - \phi_n) \;. \qquad (8.2;3)$$

Usually there exists no closed form expressions for the elements a_m.

8.3 Arbitrary slots on a cylinder

This problem is a generalization of that treated in Sec. 5. Again we have N slots each of angular width $2\psi_o$, but now their positions on the circumference are arbitrary. We suppose the centre line of the s[th] slot is at $\phi = \phi_s$, with the initial slot at $\phi = \phi_o = 0$. Then the driving field is given by

$$E_\phi(a,\phi) = E_s, \;\; \phi_s - \psi_o \leqslant \phi \leqslant \phi_s + \psi_o, \;\; s = 0, \; 1, \; ..... N-1,$$

$$= 0 \quad \text{elsewhere} \quad . \qquad (8.3;1)$$

As before, this has the representation

$$E_\phi(a,\phi) = (-ik/\omega\epsilon_o) \sum_{-\infty}^{\infty} A_m e^{im\phi} H_m^{(1)'}(ka) \quad , \tag{8.3;2}$$

so that

$$a_m = G_m^{-1} \sum_{s=0}^{N-1} E_s e^{-im\phi_s} \tag{8.3;3}$$

where $a_m = A_m e^{-im\pi/2}$ and

$$G_m = \frac{-ik}{\omega\epsilon_o} \frac{m\pi}{\sin m\psi_o} e^{im\pi/2} H_m^{(1)'}(ka) \quad . \tag{8.3;4}$$

We have

$$S_i = (-\pi ka/\omega\epsilon_o) \sum_{-\infty}^{\infty} |a_m|^2 Z_m \; , \tag{8.3;5}$$

$$S_r = (2/\omega\epsilon_o) \sum_{-\infty}^{\infty} |a_m|^2 \; . \tag{8.3;6}$$

Again we want to minimize

$$\epsilon = \sum_{-\infty}^{\infty} |a_m - \hat{a}_m|^2 + \mu \sum_{-\infty}^{\infty} (Q + \tfrac{1}{2}\pi ka Z_m) |a_m|^2 \tag{8.3;7}$$

after (3) has been inserted. This leads to

$$\sum_{s'=0}^{N-1} E_{s'} B(s',s) = Y(s) \; , \tag{8.3;8}$$

where

$$B(s,s') = \sum_{-\infty}^{\infty} [1 + \mu(Q + \tfrac{1}{2}\pi ka Z_m)] \cdot |G_m|^{-2} \cdot e^{im(\phi_s - \phi_{s'})}, \tag{8.3;9}$$

$$Y(s) = \sum_{-\infty}^{\infty} \hat{a}_m G_m^{*-1} e^{im\phi_s} \quad . \tag{8.3;10}$$

The inversion of (8) involves difficulties similar to those of Sec. 8.1. We write (8) in the matrix form

$$\mathbf{B}\,\mathbf{E} = \mathbf{Y} \tag{8.3;11}$$

where

$$\mathbf{B} = \mathbf{B}_1 + \mu \mathbf{B}_2 \quad . \tag{8.3;12}$$

To reduce (11) to the form (8.1;9), we compute $\mathbf{B}_2^{-1}$ and multiply by it on the left, giving*

$$(\mathbf{B}_2^{-1}\,\mathbf{B}_1 + \mu\,\mathbf{I})\,\mathbf{E} = \mathbf{B}_2^{-1}\,Y, \tag{8.3;13}$$

or alternatively

$$(\mathbf{B}' + \mu\,\mathbf{I})\,\mathbf{E} = Y'. \tag{8.3;14}$$

Eigenvectors $\mathbf{x}_i$ and eigenvalues λ_i are now found which satisfy $\mathbf{B}'\mathbf{x}_i = \lambda_i \mathbf{x}_i$. Then with

$$\mathbf{E} = \sum_{i=0}^{N-1} c_i\,\mathbf{x}_i\,, \tag{8.3;15}$$

$$\mathbf{Y}' = \sum_{i=0}^{N-1} d_i\,\mathbf{x}_i\,, \tag{8.3;16}$$

the solution is given by

$$c_i = d_i(\lambda_i + \mu)^{-1}, \tag{8.3;17}$$

and the parameter μ can be found in the usual manner.

IX FINAL WORDS

We have solved a number of synthesis problems in closed form, making a significant increase to the total of such problems. Our original conjecture that all problems with apertures coinciding with a complete coordinate surface are exactly solvable if the corresponding diffraction problem is, was found not true in general. Two problems – the vector problem of a circular aperture in a plane and the scalar one for a parabolic aperture – were found to be intractable. In each case, the characteristic functions did not possess sufficient orthogonality to allow a simple evaluation of the power. Nevertheless, the conjecture led us to a large number of solvable problems.

In the realm of implicit solutions, such as considered in Sec. 8, it would seem that much more can be done. For many such problems the Q definition of Uzsoky and Solymar (1957), as extended by Lo *et al* (1966), seems preferable. To adapt this definition to aperture antennas, one need only follow Lo's suggestion and associate the 'currents' J_m with the modal amplitudes in the aperture. For instance, the field in a rectangular aperture could be expanded in the form $\sum_m \sum_n A_{mn}\, e^{-2\pi inx/a} \cdot e^{-2\pi imy/b}$ and the A_{mn} treated as the J's. Evidently many interesting and practical problems can be so treated. It must be remembered, though, that computer time limits this method to apertures that are not too large.

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* We must assume $\mathbf{B}_2$ non-singular. This is probably always true.

Computer limitations and analytic difficulties encountered in some of the problems mentioned in Sec. 1.1 would seem to call for new methods. In this connection it is worth noting that synthesis is a form of optimization problem. Such problems are under intense investigation in other fields (economics, logistics, etc.). The massive literature on this subject would surely merit study by antenna engineers.

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