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A COMPUTER ORIENTED ADAPTION OF SALZER'S METHOD FOR INVERTING LAPLACE TRANSFORMS

By C. J. Shirtliffe and D. G. Stephenson

Description of problem. Salzer's method [1] for numerically inverting a Laplace transform consists of fitting a Lagrange polynomial in $1/p$ to the transform of the function and inverting the polynomial term by term. The polynomial is fitted to the values of the transform at equal intervals along the positive real axis of the complex plane.

The calculation consists of summing a series of products of interpolation coefficients and values of the transform. It is only necessary to evaluate the transform with p equal to 1, 2, 3, $\cdots$ m; where m is one less than the order of the Lagrange polynomial used.

The difficulty with this approach is that it is not possible to represent accurately every function by a polynomial which has poles only at the origin. Such a polynomial inverts to a power series which always approaches infinity as the independent variable approaches infinity. Salzer's paper contains an expression which will give an estimate of the error in the numerical inverse, but it requires the computist to make an estimate of the m^{th} derivative of $F(p)$ with respect to $1/p$. The method which follows avoids the need for this estimate of a derivative and also indicates the optimum order of Lagrange polynomial to use. The existence of an optimum m was not mentioned by Salzer.

Salzer's expression for the inverse transform is

$$L^{-1}\{F(p)\} = f(t) = \textstyle\sum_{k=1}^{m} A_k^m(t)\cdot F(k) \tag{1}$$

where

$$A_k^m(t) = \frac{k^{m+1}}{m!}\sum_{j=1}^{m} a_j \frac{t^{m-j}}{(m-j)!} \tag{2}$$

and a_j is one of a set of constant coefficients of the $m + 1$ point Lagrange interpolation polynomial coefficients. It is a function of k and is an exact integer [2]. The values of $A_k^m(t)$ are tabulated in Salzer's paper for specific values of t.

A program has been prepared to allow a digital computer to evaluate the inverse transform for any value of t, using up to an 11-point Lagrange polynomial. The inversion is done using the following double series.

$$f(t) = \textstyle\sum_{k=1}^{m} \left(\sum_{i=0}^{m-1} b_i t^i\right) F(k) \tag{3}$$

where b_i are a special set of constants derived from a_j by

$$b_i = \frac{k^{m+1}}{m!}\frac{a_{m-1}}{i!} \tag{4}$$

The values of b_i for $m = 2$ to 10 are given in Table I. The Bendix computer users program [3] has these constants incorporated with the program on punched paper tape. The values of b_i have been checked against Salzer's tables by

m	$i \backslash k$	1		2		3		4		5	
10	0	−.275573192240	..05	.253968253968	..01	−.585803571429	1	.242725925926	3	−.339084201389	4
	1	.148809523810	..03	−.134603174603	1	.304617857143	3	−.123790222222	5	.169542100694	6
	2	−.174437830688	..02	.154158730159	2	−.340937678571	4	.135441066667	6	−.181410047743	7
	3	.775462962963	..02	−.665481481481	2	.143111812500	5	−.553657837037	6	.723379629630	7
	4	−.161771797840	..01	.133681481481	3	−.277766085937	5	.104199205926	7	−.132487261737	8
	5	.174797453704	..01	−.137438518519	3	.273696215625	5	−.991007478519	6	.122406571000	8
	6	−.101646856016	..01	.747137018871	2	−.141158808482	5	.491241539424	6	−.589149380908	7
	7	.314598293021	..02	−.210291156463	2	.372477156888	4	−.124291277884	6	.144958496094	7
	8	−.478114745528	..03	.277596371882	1	−.456159661990	3	.146307386243	5	−.166563003782	6
	9	.275573192240	..04	−.126984126984		.195267857143	2	−.606814814815	3	.678168402778	4
9	0	.248015873016	..04	−.101587301587		.136687500000	2	−.364088888889	3	.339084201389	4
	1	−.109126984127	02	.436825396825	1	−.574087500000	3	.149276444444	5	−.135633680556	6
	2	.102430555556	..01	−.398222222222	2	.508477500000	4	−.128523377778	6	.113593207465	7
	3	−.356481481481	..01	.133451851852	3	−.164435062500	5	.402075496296	6	−.344735604745	7
	4	.564742476852	..01	−.201096296296	3	.237033210937	5	−.557799348148	6	.463033605505	7
	5	−.443692129630	..01	.147561481481	3	−.164558081250	5	.370012521481	6	−.207998498987	7
	6	.175334821429	..01	−.529193650794	2	.551070843750	4	−.118674773333	6	.924852159288	6
	7	−.326601473923	..02	.851736961451	1	−.818693035714	3	.169015263492	5	−.128367590526	6
	8	.223214285714	..03	−.457142857143		.410062500000	2	−.819200000000	3	.610351562500	4
8	0	−.198412698413	..03	.355555555556		−.273375000000	2	.455111111111	3	−.271267361111	4
	1	.694444444444	..02	−.120888888889	2	.902137500000	3	−.145635555556	5	.840928819444	5
	2	−.506944444444	..01	.849777777778	2	−.610993125000	4	.951182222222	5	−.530327690972	6
	3	.133101851852		−.212148148148	3	.145572187500	5	−.217239703704	6	.116690176505	7
	4	−.152314814815		.226503703704	3	−.146529000000	5	.208459851852	6	−.107873987269	7
	5	.807870370370	..01	−.108758518519	3	.653639625000	4	−.884129185185	5	.442256221065	6
	6	−.190873015873	..01	.220800000000	2	−.121682250000	4	.157240888889	5	−.764973958333	5
	7	.158730158730	..02	−.142222222222	1	.729000000000	2	−.910222222222	3	.434027777778	4
7	0	.138888888889	..02	−.106666666667	1	.455625000000	2	−.455111111111	3	.162760416667	4
	1	−.375000000000	..01	.277333333333	2	−.113906250000	4	.109226666667	5	−.374348958333	5
	2	.204861111111		−.144000000000	3	.562696875000	4	−.514275555556	5	.168457031250	6
	3	−.385416666667		.252444444444	3	−.925678125000	4	.800995555556	5	−.250922309028	6
	4	.295370370370		−.174622222222	3	.590793750000	4	−.482607407407	5	.145399305556	6
	5	−.929166666667	..01	.468800000000	2	−.144129375000	4	.111957333333	5	−.327148437500	5
	6	.972222222222	..02	−.373333333333	1	.106312500000	3	−.796444444444	3	.227864583333	4
6	0	−.833333333333	..02	.266666666667	1	−.607500000000	2	.341333333333	3	−.651041666667	3
	1	.166666666667		−.506666666667	2	.109350000000	4	−.580266666667	4	.104166666667	5
	2	−.645833333333		.182666666667	3	−.367537500000	4	.182613333333	5	−.309244791667	5
	3	.805555555556		−.204888888889	3	.376650000000	4	−.174648888889	5	.282118055556	5
	4	−.362500000000		.780000000000	2	−.128587500000	4	.563200000000	4	−.878906250000	4
	5	.500000000000	..01	−.800000000000	1	.121500000000	3	−.512000000000	3	.781250000000	3
5	0	.416666666667	..01	−.533333333333	1	.607500000000	2	−.170666666667	3	.130208333333	3
	1	−.583333333333		.693333333333	2	−.729000000000	3	.187733333333	4	−.130208333333	4
	2	.147916666667	1	−.157333333333	3	.148837500000	4	−.349866666667	4	.227864583333	4
	3	−.106944444444	1	.951111111111	2	−.789750000000	3	.173511111111	4	−.108506944444	4
	4	.208333333333		−.133333333333	2	.101250000000	3	−.213333333333	3	.130208333333	3
4	0	−.166666666667		.800000000000	1	−.405000000000	2	.426666666667	2		
	1	.150000000000	1	−.640000000000	2	.283500000000	3	−.256000000000	3		
	2	−.216666666667	1	.760000000000	2	−.283500000000	3	.234666666667	3		
	3	.666666666667		−.160000000000	2	.540000000000	2	−.426666666667	2		
3	0	.500000000000		−.800000000000	1	.135000000000	2				
	1	−.250000000000	1	.320000000000	2	−.405000000000	2				
	2	.150000000000	1	−.120000000000	2	.135000000000	2				
2	0	−.100000000000	1	.400000000000	1						
	1	.200000000000	1	−.400000000000	1						

NOTE. The numbers in the table are in a floating point form; the number at the right is the characteristic. The format is of places to move the decimal to the left.

LE I

ical Calculation of Inverse Laplace Transforms

6		7		8		9		10	
.209952000000	5	−.653877891204	5	.106522006349	6	−.864777877232	5	.275573192240	5
−.102876480000	7	.313861387778	7	−.500653429841	7	.397797823527	7	−.124007936508	7
.107705376000	8	−.321707922472	8	.502783869968	8	−.391744378386	8	.119874338624	8
−.419694048000	8	.122732880179	9	−.188153370548	9	.144071994347	9	−.434027777778	8
.750657132000	8	−.215068611891	9	.323955613393	9	−.244332179488	9	.726514274691	8
−.677446869600	8	.190432127645	9	−.282410255550	9	.210266416960	9	−.618489583333	8
.318929918400	8	−.881423764688	8	.128978324760	9	−.950011345676	8	.276981677445	8
−.769472413714	7	.209605228582	8	−.303336630509	8	.221483340992	8	−.641199766944	7
.869926114286	6	−.234228401742	7	.336019864671	7	−.243682051834	7	.701629031242	6
−.349920000000	5	.934111273148	5	−.133152507937	6	.960864308036	5	−.275573192240	5
−.139968000000	5	.280233381944	5	−.266305015873	5	.960864308036	4		
.545875200000	6	−.106488685139	7	.985328558730	6	−.345911150893	6		
−.445098240000	7	.846304813472	7	−.764295395556	7	.262315956094	7		
.131420952000	8	−.243896453419	8	.215618294519	8	−.726413416875	7		
−.171863208000	8	.311981488841	8	−.270843297185	8	.898768452129	7		
.107904830400	8	−.192174712225	8	.164339044504	8	−.538756617516	7		
−.327785616000	7	.574618549677	7	−.485474043937	7	.157640466003	7		
.447164434286	6	−.774244800972	6	.648072277914	6	−.208919353833	6		
−.209952000000	5	.360300062500	5	−.299593142857	5	.960864308036	4		
.699840000000	4	−.800666805556	4	.332881269841	4				
−.209952000000	6	.232193373611	6	−.932067555556	5				
.128070720000	7	−.137314357153	7	.535938844444	6				
−.272937600000	7	.284903938310	7	−.108741214815	7				
.245206440000	7	−.250341821204	7	.938863881481	6				
−.981525600000	6	.984553281898	6	−.364283069630	6				
.166639680000	6	−.164937361944	6	.604179504762	5				
−.933120000000	4	.915047777778	4	−.332881269841	4				
−.233280000000	4	.114380972222	4						
.513216000000	5	−.240200041667	5						
−.221616000000	6	.100083350694	6						
.318816000000	6	−.140116690972	6						
−.179722800000	6	.773977912037	5						
.396187200000	5	−.168140029167	5						
−.272160000000	4	.114380972222	4						
.388800000000	3								
−.583200000000	4								
.165240000000	5								
−.145800000000	5								
.443880000000	4								
−.388800000000	3								

±.DDDDDDDDDDDD XX where XX is the number of places to move the decimal to the right and, ..XX is the number evaluating

$$\textstyle\sum_{i=0}^{m-1} b_i = A_k^m(1)$$

The values of $A_k^m(1)$ thus calculated have more significant figures than are given in Salzer's paper but when they are rounded to the same number of digits the agreement is perfect.

The values of $f(t)$ determined using different values of m should be the same (independent of m) so long as they are correct. Thus a comparison of the results of a series of calculations with successive values of m indicates the maximum value of t for which the value of $f(t)$ is accurate. The results also show the optimum value of m.

The computer has been used to invert the following transforms for several values of the pole positions:

$$\frac{1}{p}, \quad \frac{1}{p^2}, \quad \frac{1}{p+c}, \quad \frac{1}{(p+c)^2}, \quad \frac{1}{p^2+c^2}, \quad \frac{1}{(p+c)^2+d^2}$$

These transforms were studied because of the position of their poles and because the exact values of the inverse transforms were readily available.

Results. It was found that increasing the order of the Lagrange polynomial (i.e. the value of m) did not necessarily increase the range or accuracy of the

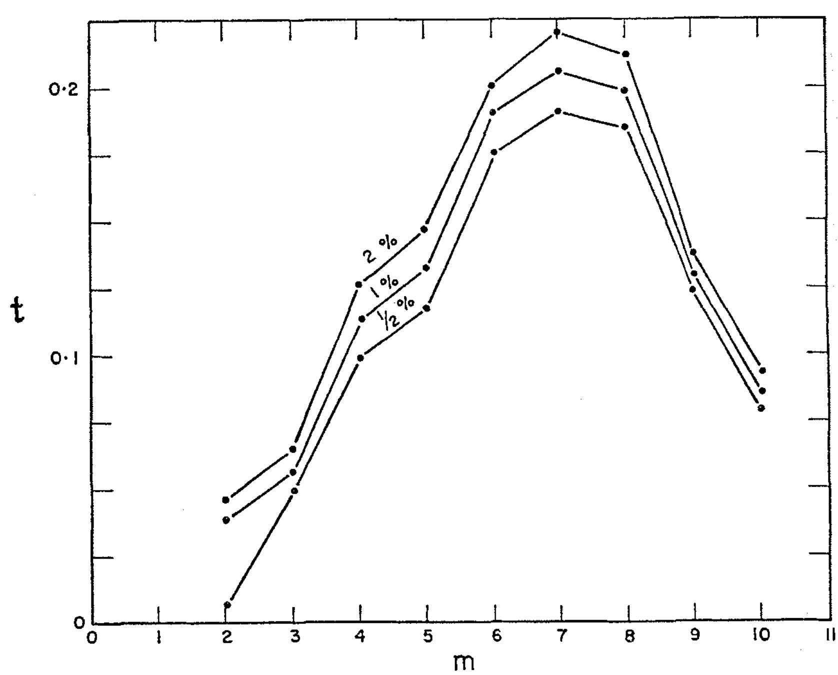

FIG. 1. Precision of numerical inversion of $\frac{1}{p^2 + (0.1)^2}$

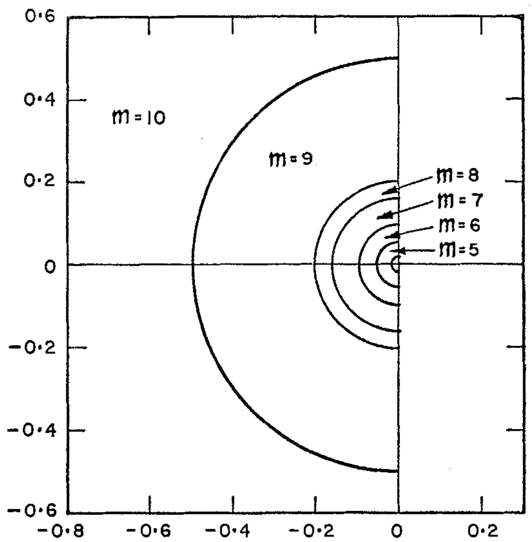

FIG. 2. Dependence of optimum m on position of poles in p-plane

inverse transform. For each function there is an optimum m which will give the least error in $f(t)$ over the greatest range of t. For exact polynomials in $1/p$ the optimum m obviously corresponds to the highest power of $1/p$. When $F(p)$ is not an exact polynomial in $1/p$ the optimum value of m seems to depend primarily on the position of the poles of $F(p)$.

Figure 1 shows a typical variation in error with m and t for poles at $\pm i(0.1)$. By making similar evaluations for the inversions of $1/(p + b)$, $1/(p^2 + b^2)$, and $1/[(p + c)^2 + d^2]$ it was possible to establish an approximate relationship between the pole position and the optimum m. This is shown graphically in Fig. 2. This relationship is of no practical value in finding the optimum m since the poles of $F(p)$ are not usually known.

The optimum m and the upper limit of t can both be found by repeated application of the numerical inversion using successive values of m. In this way the results which agree to the highest t for three consecutive values of m can be found. The optimum value is the middle one of the three. Figure 3 shows the values of $f(t)$ calculated using values of m from 4 to 9, for $F(p) = 1/(p^2 + (0.1)^2)$. The optimum m is seen to be seven. The error in the inversion obtained with the optimum m is less than half the difference between it and the values obtained when m is one larger and one smaller than the optimum. Therefore, once the optimum m has been found the limits on the error are indicated without any further calculation.

The method for finding the optimum m only works when the optimum is less than 10 since only the 3 to 11 point interpolation formulas are used. It is possible to distinguish cases where the optimum m is 10 or only slightly greater from cases where it is much greater than 10 by examining the difference in the values of the inversion at $t = 0$ for $m = 9$ and $m = 10$. If the difference is large, the

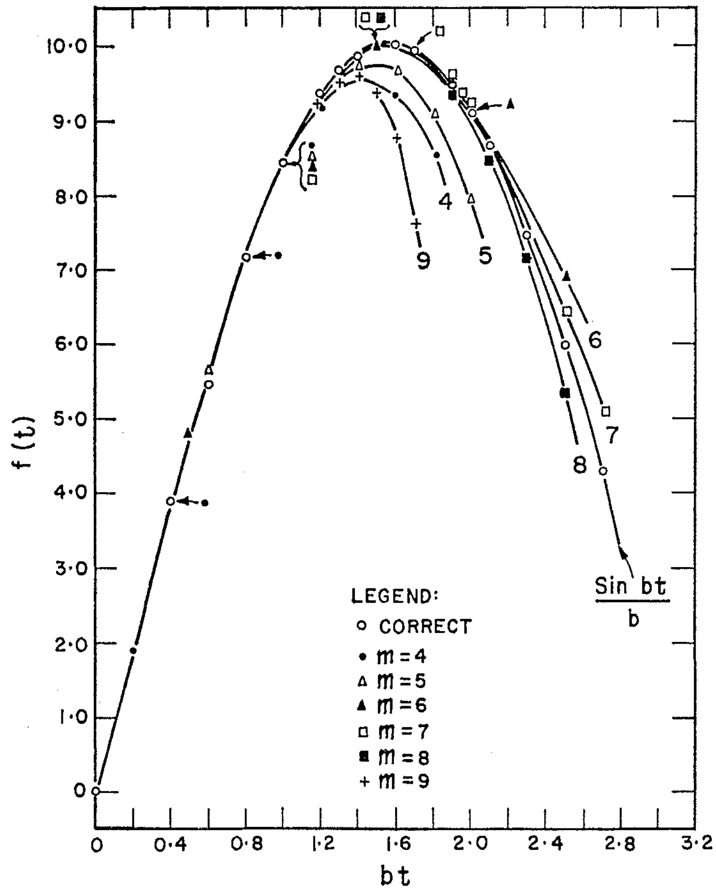

FIG. 3. $L^{-1}\left(\frac{1}{p^2+b^2}\right)$ versus "bt" for $m = 4$ to 9, $b = 0.1$

optimum m is considerably greater than 10 and the method cannot be used. If the difference is less than the allowable error, the optimum m is 10 or near enough to 10 that the method can be used. The limit on the error cannot be found by the usual method in such cases. If the values for a particular t at $m = 9$ and $m = 10$ agree within the allowable error, however, the error in the inversion using $m = 10$ will not exceed the allowable error.

It would be possible to calculate the Lagrange interpolation constants for m's higher than 10. The greater the m, however, the greater the number of significant figures lost in the summation. The number lost can exceed 12 using $m = 10$ and $t = 10$. The maximum usable m would therefore depend on the computer used for the calculations. The transform must be defined to more figures than the number of significant figures dropped in the calculation, however, there is no advantage in having the transform defined to more figures than the constants.

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