Supplemental material:
Robust quantum metrology with random Majorana constellations

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RANDOM STATES FROM SU$(2S+1)$

We would like to compute the following integrals for random state coefficients $\psi_m = a_m + \mathrm{i} b_m$:

$$I_{m,m',q} = \int \psi_{m+q} \psi_m^* \psi_{m'+q}^* \psi_{m'} \, dU. \tag{1}$$

These integrals and more are calculated in Ref. [1], where we can use their Eq. (35) to calculate

$$\begin{aligned} \int |\psi_m|^2 |\psi_n|^2 dU &= \frac{\Gamma(2S+1)}{\Gamma(2S+1+2)} 1!1!, \quad m \neq n, \\ \int |\psi_m|^4 dU &= \frac{\Gamma(2S+1)}{\Gamma(2S+1+2)} 2!. \end{aligned} \tag{2}$$

The asymptotic behavior of such integrals was studied by Weingarten [2].

An alternative route is to explicitly compute the integrals using the normalized Haar measure for SU$(2S+1)$:

$$\begin{aligned} I_{m,m',q} = & \frac{1}{\pi^{2S+1}} \int (a_{m+q} + \mathrm{i} b_{m+q})(a_m - \mathrm{i} b_m)(a_{m'+q} - \mathrm{i} b_{m'+q})(a_{m'} + \mathrm{i} b_{m'}) \\ & \times \frac{\exp(-a_{-S}^2 - b_{-S}^2 - \cdots - a_S^2 - b_S^2)}{(a_{-S}^2 + b_{-S}^2 + \cdots + a_S^2 + b_S^2)^2} da_{-S} db_{-S} \cdots da_S db_S. \end{aligned} \tag{3}$$

The integrals vanish unless $m+q=m$ and $m'+q=m'$ or $m+q=m'+q$ and $m=m'$, because the integrand is otherwise an odd function of some coefficients a_m and b_m, so we only need consider

$$\begin{aligned} I_{m,m',0} = & \frac{1}{\pi^{2S+1}} \int (a_m^2 + b_m^2)(a_{m'}^2 + b_{m'}^2) \frac{\exp(-a_m^2 - b_m^2 - a_{m'}^2 - b_{m'}^2 - \mathbf{r}^2)}{(a_m^2 + b_m^2 + a_{m'}^2 + b_{m'}^2 + \mathbf{r}^2)^2} da_m db_m da_{m'} db_{m'} d^{4S-2}\mathbf{r} \\ = & \frac{1}{\pi^{2S+1}} \int \mathbf{r}_1^2 \mathbf{r}_2^2 \frac{\exp(-\mathbf{r}_1^2 - \mathbf{r}_1^2 - \mathbf{r}^2)}{(\mathbf{r}_1^2 + \mathbf{r}_2^2 + \mathbf{r}^2)^2} d^2\mathbf{r}_1 d^2\mathbf{r}_2 d^{4S-2}\mathbf{r}, \\ I_{m,m,q} = & \frac{1}{\pi^{2S+1}} \int (a_{m+q}^2 + b_{m+q}^2)(a_m^2 + b_m^2) \frac{\exp(-a_{m+q}^2 - b_{m+q}^2 - a_m^2 - b_m^2 - \mathbf{r}^2)}{(a_{m+q}^2 + b_{m+q}^2 + a_m^2 + b_m^2 + \mathbf{r}^2)^2} da_{m+q} db_{m+q} da_m db_m d^{4S-2}\mathbf{r} = I_{m,m',0}, \\ I_{m,m,0} = & \frac{1}{\pi^{2S+1}} \int \mathbf{r}_1^4 \frac{\exp(-\mathbf{r}_1^2 - \mathbf{r}^2)}{(\mathbf{r}_1^2 + \mathbf{r}^2)^2} d^2\mathbf{r}_1 d^{4S}\mathbf{r}, \end{aligned} \tag{4}$$

where we have used the vectors $\mathbf{r}$ to encompass all of the remaining coordinates. Since these integrals all take the form $I = \int |\psi_m|^2 |\psi_n|^2 dU$, it is tempting to use Isserlis's theorem to express them in terms of $\int |\psi_m|^2 dU$, which is easier to calculate; however, the coefficients ψ_m are not themselves normal random variables because they must satisfy normalization constraints. Instead, we can only use the moments-cumulants formula to guarantee that

$$I_{m,m,0} = 2 I_{m,m',0}, \quad m \neq m', \tag{5}$$

thus justifying our statement in the main text that

$$I_{m,m',q} \propto \delta_{q,0} + \delta_{m,m'}. \tag{6}$$

We can also directly calculate these volume integrals. The angular dependence can easily be factored out, using

$$\int d^n\mathbf{r} = \frac{2\pi^{n/2}}{\Gamma(n/2)}\int_0^\infty r^{n-1}dr, \tag{7}$$

leading to

$$\begin{aligned} I_{m,m',0} =& \frac{2^3}{\Gamma(2S-1)}\int_0^\infty r_1^2 r_2^2 \frac{\exp(-r_1^2-r_2^2-r^2)}{(r_1^2+r_2^2+r^2)^2} r_1 r_2 r^{4S-3} dr_1 dr_2 dr \\ =& \frac{2^3}{\Gamma(2S-1)}\int_0^{\pi/2}\cos^3\theta\sin^3\theta d\theta\int_0^{\pi/2}\cos^7\phi\sin^{4S-3}\phi d\phi\int_0^\infty x^{4S+1}\mathrm{e}^{-x^2}dx \\ =& \frac{1}{(2S+1)(2S+2)} \end{aligned} \tag{8}$$

and

$$\begin{aligned} I_{m,m,0} =& \frac{2^2}{\Gamma(2S)}\int_0^\infty r_1^5 r^{4S-1}\frac{\exp(-r_1^2-r^2)}{(r_1^2+r^2)^2}dr_1 dr \\ =& \frac{2^2}{\Gamma(2S)}\int_0^{\pi/2}\cos^5\theta\sin^{4S-1}\theta d\theta\int_0^\infty x^{4S+1}\mathrm{e}^{-x^2}dx \\ =& \frac{2}{(2S+1)(2S+2)}. \end{aligned} \tag{9}$$

This yields the result given in Eq. (2).

Since $\sum_{m=-S}^{S} C_{Sm,Kq}^{Sm+q}$ vanishes for $q=0$, we only need to further consider the $I_{m,m,q}$ terms to calculate the averaged multipoles. We then verify the final result of Eq. (8).

$$\mathcal{A}_M^{(S)} = \frac{1}{(2S+1)(2S+2)}\sum_{K=1}^{M}\sum_{q=-K}^{K}\frac{2K+1}{2S+1}\sum_{m=-S}^{S}\left(C_{Sm,Kq}^{Sm+q}\right)^2 = \frac{1}{(2S+1)(2S+2)}\sum_{K=1}^{M}(2K+1) = \frac{M(M+2)}{(2S+1)(2S+2)}. \tag{10}$$

SYMMETRIC PROJECTIONS OF $2S$ RANDOM QUBITS

Random Majorana constellations are intimately related to the symmetrized tensor products of $2S$ random qubits, albeit with profound differences. To start, we can relate the coefficients of a state given in the angular momentum basis to the coordinates of the Majorana constellation through

$$\psi_k = \sqrt{\frac{(S+k)!(S-k)!}{\mathcal{N}}^2}\sum_{\text{perm}}\prod_{i=j_1}^{j_{S+k}}\cos\left(\frac{\theta_i}{2}\right)\prod_{i=j_{S+k+1}}^{j_{2S}}\sin\left(\frac{\theta_i}{2}\right)e^{i\phi_i}, \tag{11}$$

where the sum is over all symmetric permutations $\mathbf{j}$ of the integers from 1 to $2S$. Here, $\mathcal{N}$ is equal to the permanent of the matrix M with elements [3]

$$M_{ij} = \cos\left(\frac{\theta_i}{2}\right)\cos\left(\frac{\theta_j}{2}\right) + \sin\left(\frac{\theta_i}{2}\right)\sin\left(\frac{\theta_j}{2}\right)e^{i(\phi_i-\phi_j)}. \tag{12}$$

Next, we consider the state of $2S$ qubits projected onto the symmetric subspace. The translation between the two pictures is given by

$$|S,m\rangle = \frac{1}{\sqrt{\binom{2S}{S+m}}}\sum_{\text{perm}}|+\rangle^{\otimes S+m}\otimes|-\rangle^{\otimes S-m}, \tag{13}$$

facilitating the projection operator $\mathcal{P}_S = \sum_{m=-S}^{S}|S,m\rangle\langle S,m|$ onto the symmetric subspace. A symmetric state can always be composed from a symmetrized $\mathrm{SU}(2)^{\otimes 2S}$ rotation operator, which we remark is no longer a unitary operator,

$$U_{\mathcal{P}_S} = \mathcal{P}_S U^{(1)}\otimes\cdots\otimes U^{(2S)}\mathcal{P}_S \tag{14}$$

acting on the fiducial state $|+\rangle^{\otimes 2S}$, where

$$U^{(i)}\left|+\right\rangle_i = u^{(i)}_{00}\left|+\right\rangle_i + u^{(i)}_{10}\left|-\right\rangle_i = \cos\frac{\theta_i}{2}\left|+\right\rangle_i + \sin\frac{\theta_i}{2}\mathrm{e}^{\mathrm{i}\phi_i}\left|-\right\rangle_i . \tag{15}$$

In fact, this process immediately yields

$$\langle S,k|\, U_{\Pi_S}\left|+\right\rangle^{\otimes 2S} = \binom{2S}{S+k}^{-1/2}\sum_{\text{perm}}\prod_{i=j_1}^{j_{S+k}}\cos\left(\frac{\theta_i}{2}\right)\prod_{i=j_{S+k+1}}^{j_{2S}}\sin\left(\frac{\theta_i}{2}\right)e^{i\phi_i}, \tag{16}$$

which is equivalent to ψ_k found in Eq. S(11) up to replacing the normalization factor by $\mathcal{N}\to\sqrt{(2S)!}$. This crucial difference of the constellation-dependent normalization factor significantly changes the properties of random Majorana constellations versus symmetrized projections of random qubits. In fact, it is the absence of the normalization constant in the latter case that makes the present integrals analytically tractable, which reinforces the need to present numerical studies of random Majorana constellations.

We can directly calculate the multipoles from the symmetrized projection states because $|+\rangle^{\otimes 2S}$ and T_{Kq} are already symmetric:

$$\begin{aligned}\varrho_{Kq} &= \langle +|^{\otimes 2S}\, U^{\dagger}_{\Pi_S} T^{\dagger}_{Kq} U_{\Pi_S}\left|+\right\rangle^{\otimes 2S} = \langle +|^{\otimes 2S}\, U^{(1)\dagger}\otimes\cdots\otimes U^{(2S)\dagger} T^{\dagger}_{Kq} U^{(1)}\otimes\cdots\otimes U^{(2S)}\left|+\right\rangle^{\otimes 2S} \\ &= \sum_{\mathbf{i},\mathbf{j}=\mathbf{0}}^{\mathbf{1}} u^{(1)}_{i_1 0}\cdots u^{(2S)}_{i_{2S}0}\,\langle\mathbf{i}|\, T^{\dagger}_{Kq}\,|\mathbf{j}\rangle\, u^{(1)*}_{j_1 0}\cdots u^{(2S)*}_{j_{2S}0},\end{aligned} \tag{17}$$

where the sums run over all 2^{2S} bit strings of length $2S$ and $|\mathbf{i}\rangle = |i_1\rangle_1\otimes\cdots\otimes|i_{2S}\rangle_{2S}$, with the convenient translation $(|0\rangle,|1\rangle)\to(|+\rangle,|-\rangle)$.

Averaging over the squares of the multipoles is straightforward because, for all $m\in 1,\cdots,2S$,

$$\int u^{(m)}_{i0}u^{(m)*}_{j0}u^{(m)*}_{k0}u^{(m)}_{l0}\,dU^{(m)} = \frac{\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}}{6}. \tag{18}$$

This lets us express the average squared multipole as

$$\int\left|\varrho_{Kq}\right|^2 dU = \sum_{\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l}=\mathbf{0}}^{\mathbf{1}}\langle\mathbf{i}|\, T^{\dagger}_{Kq}\,|\mathbf{j}\rangle\langle\mathbf{l}|\, T_{Kq}\,|\mathbf{k}\rangle\prod_{m=1}^{2S}\frac{\delta_{i_m j_m}\delta_{k_m l_m}+\delta_{i_m k_m}\delta_{j_m l_m}}{6}. \tag{19}$$

Then, noting that each state of the form $|\mathbf{i}\rangle$ is not symmetrized and writing $|\mathbf{i}| = \sum_{m=1}^{2S} i_m$, we can calculate expressions like

$$\langle\mathbf{l}|\, T_{Kq}\,|\mathbf{k}\rangle = \sqrt{\frac{2K+1}{2S+1}}\left[\binom{2S}{|\mathbf{l}|}\binom{2S}{|\mathbf{k}|}\right]^{-1/2} C^{S,S-|\mathbf{l}|}_{S,S-|\mathbf{k}|,Kq}, \tag{20}$$

which set the constraints $|\mathbf{l}| = |\mathbf{k}| - q$. This leads to the closed-form expression

$$\int\left|\varrho_{Kq}\right|^2 dU = \frac{2K+1}{2S+1}\sum_{\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l}=\mathbf{0}}^{\mathbf{1}}\delta_{|\mathbf{k}|,|\mathbf{l}|+q}\delta_{|\mathbf{i}|,|\mathbf{j}|+q}\left[\binom{2S}{|\mathbf{j}|}\binom{2S}{|\mathbf{i}|}\binom{2S}{|\mathbf{l}|}\binom{2S}{|\mathbf{k}|}\right]^{-1/2} C^{S,S-|\mathbf{l}|}_{S,S-|\mathbf{k}|,Kq}C^{S,S-|\mathbf{j}|}_{S,S-|\mathbf{i}|,Kq}\prod_{m=1}^{2S}\frac{\delta_{i_m j_m}\delta_{k_m l_m}+\delta_{i_m k_m}\delta_{j_m l_m}}{6}. \tag{21}$$

As it stands, Eq. (21) can be used to exactly compute all of the averaged multipoles. However, the number of terms in the sum grows exponentially with S, making the expression rather cumbersome, so we can invoke a number of counting arguments to simplify the expressions. The terms contributing to the sum must be of the form

$$(i_m,j_m,k_m,l_m)\in\{(0,0,0,0),(0,0,1,1),(1,1,0,0),(1,1,1,1),(0,1,0,1),(1,0,1,0)\} \tag{22}$$

and we must partition the sums by counting how many such terms are present, because we find an extra factor of 2 for every index m with $i_m = j_m = k_m = l_m$. Denoting these numbers of terms by $n_1,\cdots,n_6$, respectively, we have the following constraints:

$$\begin{aligned} &n_1+n_2+n_3+n_4+n_5+n_6 = 2S \\ &n_3+n_4+n_6 = |\mathbf{i}| \\ &n_3+n_4+n_5 = |\mathbf{j}| = |\mathbf{i}| - q \\ &n_2+n_4+n_6 = |\mathbf{k}| \\ &n_2+n_4+n_5 = |\mathbf{l}| = |\mathbf{k}| - q.\end{aligned} \tag{23}$$

These imply that $n_6 = n_5 + q$ and $n_3 = n_2 + |\mathbf{i}| - |\mathbf{k}|$. For a given pair of $|\mathbf{i}|$ and $|\mathbf{k}|$, we are left with two constraints on the remaining four numbers:

$$\begin{aligned} n_1 + 2n_2 + n_4 + 2n_5 &= 2S + |\mathbf{k}| - |\mathbf{i}| - q \\ n_2 + n_4 + n_5 &= |\mathbf{k}| - q. \end{aligned} \tag{24}$$

These lead to the overall sums

$$\begin{aligned} \int \left|\varrho_{Kq}\right|^2 dU = \frac{2K+1}{2S+1} \sum_{|\mathbf{i}|,|\mathbf{k}|=0}^{2S} \left[\binom{2S}{|\mathbf{i}|-q}\binom{2S}{|\mathbf{i}|}\binom{2S}{|\mathbf{k}|-q}\binom{2S}{|\mathbf{k}|}\right]^{-1/2} C^{S,S-|\mathbf{k}|+q}_{S,S-|\mathbf{k}|,Kq} C^{S,S-|\mathbf{i}|+q}_{S,S-|\mathbf{i}|,Kq} \frac{1}{6^{2S}} \\ \times \sum_{n_2,n_5} 2^{2S-|\mathbf{i}|+|\mathbf{k}|-q-2n_2-2n_5} \binom{2S}{2S-|\mathbf{i}|-n_2-n_5, n_2, n_2+|\mathbf{i}|-|\mathbf{k}|, |\mathbf{k}|-q-n_2-n_5, n_5, n_5+q}. \end{aligned} \tag{25}$$

These can be explicitly calculated by summing the $O(S^4)$ terms for each K and q. While all pure spin-S states satisfy $A_{2S}^{(S)} = 1 - A_0^{(S)} = 2S/(2S+1)$, the projection of a $2S$-qubit state will in general have $A_{2S}^{(S)} < 2S/(2S+1)$.

An alternate counting argument yields an even simpler result, reducing the number of sums by one, as follows. First, there are $\binom{2S}{|\mathbf{i}|}$ choices for the locations of the 0s and 1s in $\mathbf{i}$. Then, there are $\binom{|\mathbf{i}|}{M}$ choices for which of the locations of the 1s of $\mathbf{l}$ match those of $\mathbf{i}$ and $\binom{2S-|\mathbf{i}|}{|\mathbf{l}|-M}$ choices for the locations that do not match. Both of $\mathbf{j}$ and $\mathbf{k}$ must have 1s at the M matching locations, leaving $\binom{|\mathbf{i}|+|\mathbf{l}|-2M}{|\mathbf{j}|-M} = \binom{|\mathbf{i}|+|\mathbf{l}|-2M}{|\mathbf{k}|-M}$ choices for distributing the remaining 1s of $\mathbf{j}$ and $\mathbf{k}$ among the as-yet unpaired 1s of $\mathbf{i}$ and $\mathbf{l}$ (note: $|\mathbf{i}| + |\mathbf{l}| = |\mathbf{j}| + |\mathbf{k}|$). The extra factors of 2 arise M times from the terms with $i_m = j_m = k_m = l_m = 1$ and $2S - |\mathbf{i}| - |\mathbf{l}| + M$ times when $i_m = j_m = k_m = l_m = 0$. These considerations lead to the overall sums

$$\prod_{m=1}^{2S} \delta_{i_m j_m}\delta_{k_m l_m} + \delta_{i_m k_m}\delta_{j_m l_m} = \delta_{|\mathbf{i}|+|\mathbf{l}|,|\mathbf{j}|+|\mathbf{k}|}\binom{2S}{|\mathbf{i}|} 2^{2S-|\mathbf{i}|-|\mathbf{l}|} \sum_M \binom{|\mathbf{i}|}{M}\binom{2S-|\mathbf{i}|}{|\mathbf{l}|-M}\binom{|\mathbf{i}|+|\mathbf{l}|-2M}{|\mathbf{j}|-M} 4^M, \tag{26}$$

where the constraints on the allowed values of M are directly enforced by the binomial coefficients. Finally,

$$\begin{aligned} \int \left|\varrho_{Kq}\right|^2 dU &= \frac{2K+1}{2S+1} \sum_{i,k=0}^{2S} \left[\binom{2S}{i-q}\binom{2S}{i}\binom{2S}{k-q}\binom{2S}{k}\right]^{-1/2} C^{S,S-k+q}_{S,S-k,Kq} C^{S,S-i+q}_{S,S-i,Kq} \frac{2^{-i-k+q}}{3^{2S}}\binom{2S}{i} \\ &\quad \times \sum_M \binom{i}{M}\binom{2S-i}{k-q-M}\binom{i+k-q-2M}{i-q-M} 4^M \\ &= \frac{2K+1}{2S+1} \sum_{m,m'=-S}^{S} \left[\binom{2S}{S+m+q}\binom{2S}{S+m}\binom{2S}{S+m'+q}\binom{2S}{S+m'}\right]^{-1/2} C^{S,m'+q}_{S,m',Kq} C^{S,m+q}_{S,m,Kq} \frac{2^{m+m'+q}}{6^{2S}}\binom{2S}{S+m} \\ &\quad \times \sum_M \binom{S-m}{M}\binom{S+m}{S-m'-q-M}\binom{2S-m-m'-q-2M}{S-m-q-M} 4^M \\ &= \frac{2K+1}{2S+1} \sum_{m,m'=-S}^{S} \left[\binom{2S}{S+m+q}\binom{2S}{S+m}\binom{2S}{S+m'+q}\binom{2S}{S+m'}\right]^{-1/2} C^{S,m'+q}_{S,m',Kq} C^{S,m+q}_{S,m,Kq} \frac{2^{m+m'+q}}{6^{2S}}\binom{2S}{S+m+q} \\ &\quad \times \sum_M \binom{S-m-q}{M}\binom{S+m+q}{S-m'-M}\binom{2S-m-m'-q-2M}{S-m-M} 4^M. \end{aligned} \tag{27}$$

These averaged multipoles can now be computed by summing only $O(S^3)$ terms.

What are these averages, given that the states in question are not normalized? They are averages over the randomly projected states weighted by the factor $\mathcal{N}^2$. If the weight factor was $\mathcal{N}$, the averages would correspond to being weighted by the probability of finding a particular state; that is what is plotted in the figures in the main text. Such an average could not be computed analytically. The analytically computed averages here correspond to being scaled by the *square* of the probability of that projection succeeding, which magnifies the contributions of the states that are more likely to appear. The results can be normalized as usual using the 0th order multipole and then appear even more classical than when $\mathcal{N}$ is the weight. These show that the distribution of randomly projected states is strongly skewed toward classical states, unlike that RM states and CUE states.

AVERAGE QFI FOR CUE

The QFI for a given pure state $|\psi\rangle$ is $4(\langle\psi|(\mathbf{S}\cdot\mathbf{u})^2|\psi\rangle - \langle\psi|\mathbf{S}\cdot\mathbf{u}|\psi\rangle^2)$. Averaging this value over all rotation angles yields $4\sum_{i=1}^{3}\mathrm{Var}_\psi(J_i)/3$. What happens when $|\psi\rangle$ is a random state drawn from the CUE? Immediately, we realize that each of the three variances will be the same, so the average QFI will be $4\int dU\,\mathrm{Var}_{U\psi}(J_3)$. This quantity is readily calculable, as follows.

Take the fiducial state $|\psi\rangle = |SS\rangle$, an eigenstate of J_3. Expressing U in this basis, we observe that $U|\psi\rangle = \sum_{m=-S}^{S} u_{mS}\,|m\rangle$. The average QFI becomes

$$
\begin{aligned}
\int dU \bar{\mathsf{Q}}_{U\psi}(\theta) &= 4\sum_{m=-S}^{S} m^2 \int dU|u_{mS}|^2 - 4\sum_{m,n=-S}^{S} mn\int dU|u_{mS}|^2|u_{nS}|^2 \\
&= 4\sum_{m=-S}^{S} m^2\frac{\Gamma(2S+1)}{\Gamma(2S+2)} - 4\sum_{m,n=-S}^{S} mn\frac{\Gamma(2S+1)}{\Gamma(2S+3)}(1+\delta_{mn}) \\
&= 4\sum_{m=-S}^{S} m^2\left[\frac{\Gamma(2S+1)}{\Gamma(2S+2)} - \frac{\Gamma(2S+1)}{\Gamma(2S+1+2)}\right] \\
&= \frac{2S(2S+1)}{3}.
\end{aligned}
\tag{28}
$$

This is almost as good as the optimal value, taken by King states of at least second order, of $\bar{\mathsf{Q}} = 4S(S+1)/3$.

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