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Behaviour of a semi-infinite beam in a creeping medium

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The behaviour of a pipeline embedded in a creeping medium is examined. Approximate solutions for a beam in a creeping foundation are developed, and characteristic nondimensional load–displacement relationships are presented. A comparison of these approximate solutions provides upper and lower bound solutions that are consistent with finite element analyses. Furthermore, the simplified solutions can be readily adapted for analyzing the uplift behaviour of shallow pipelines. These solutions can also be used to analyze the creeping behaviour of laterally loaded piles. The results are presented in the form of nondimensional charts that permit hand calculations and rapid verification of the structural design of the pipelines and piles. An approximate three-dimensional solution that accounts for embedment is proposed.

Key words: creeping behaviour of pipelines, creeping foundation, laterally loaded pile.

Le comportement d'un pipeline enfoui dans un milieu en fluage est examiné. Des solutions approximatives pour une poutre dans une fondation sujette au fluage sont développées, et des relations caractéristiques adimensionnelles charge-déplacement sont présentées. Une comparaison de ces solutions approximatives fournit des solutions des limites inférieure et supérieure qui sont consistantes avec les analyses en éléments finis. De plus, les solutions simplifiées peuvent être aisément adaptées à l'analyse du comportement en soulèvement de pipelines peu profonds. Ces solutions peuvent aussi être utilisées pour analyser le comportement en fluage de pieux chargés latéralement. Les résultats sont présentés sous la forme de chartes adimensionnelles qui permettent des calculs à la main et une vérification rapide du calcul structural des pipelines et des pieux. Une solution approximative 3-dimensionnelle tenant compte de l'enfouissement est proposée.

Mots clés : comportement en fluage des pipelines, fondation en fluage, pieu chargé latéralement.

[Traduit par la rédaction]

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Introduction

The understanding of soil–pipeline interaction, particularly in the context of a frozen surrounding medium, is important for pipeline and pile design. The different aspects that need to be considered, particularly when involved with the design of pipelines, are (*i*) the mechanics of frost susceptibility and frost heave which essentially constitute the loading process, (*ii*) the modelling of mechanical properties of frozen ground, and (*iii*) the mechanical response of the pipeline. Though each of these aspects has been well studied individually, there is a lack of proper understanding of the interaction between frozen soil and pipelines subjected to uplift.

Rajani and Morgenstern (in preparation) have recently summarized the state of knowledge in each of the identified aspects related to frozen ground. The analysis of the interaction of frost heave with a pipeline is a complex problem in which many processes need to be examined for a proper understanding of the complete system. In the present work, we propose to decouple the frost heave process in frost-susceptible soil from the pipeline in the non-frost-susceptible soil. This implies that we can apply an attenuated frost heave rate at the transition zone of the two types of media rather than the free-field frost heave rate (that which is usually measured in the laboratory). This is illustrated in Fig. 1 using data from the Caen, France, experiments (Dallimore and Crawford 1984) where the pipeline is embedded in both sand and silt. The attenuation of the free-field frost heave is probably a function of the dimensions and mechanical properties of the adjacent frozen ground. Ladanyi and Lemaire (1984) attempted to back-analyze the Caen experiments using a simplified model based on the elastic Winkler foundation that accounted for free-field frost heave in an idealized manner. Here, we assume that the attenuated relation can be readily approximated from the stress dependence of the free-field frost heave rate (Konrad and Morgenstern 1982).

Previous attempts at solving this problem related to pipelines have been made by Nixon *et al.* (1983) and Selvadurai (1988). Nixon *et al.* (1983) simplified the problem to that of plane strain conditions and applied the free-field frost heave over a predetermined section of the frost-susceptible soil and studied its attenuation specifically at the interface of the frost-susceptible and non-frost-susceptible soils. However, the pipeline was considered as a passive component of the whole system, and hence its interaction effects were not studied. Selvadurai (1988) analyzed the elastic behaviour of an embedded pipeline at shallow depth using the thermoelastic analogy. Frozen soil hardly behaves as an elastic material, and hence the application of this analysis is limited.

The classical studies of Glen (1955) indicate that the flow law of ice-rich soils is that of the Norton type. The Norton creep relationship, rewritten in the generalized form as pro-

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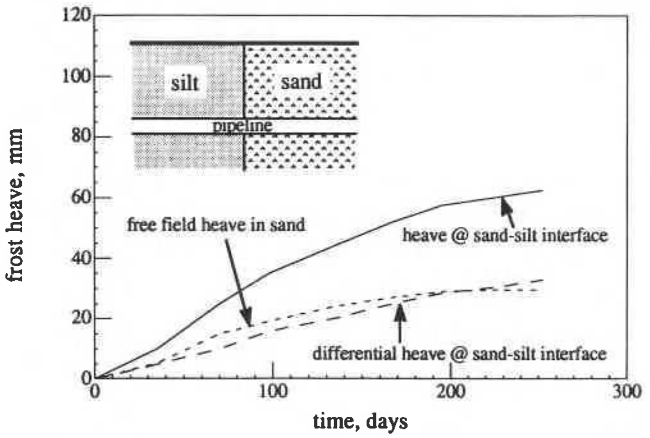

FIG. 1. Differential frost heave at the interface of discontinuous permafrost.

posed by Ladanyi (1972), is

$$[1] \quad \frac{\dot{\epsilon}}{\dot{\epsilon}_0} = \left(\frac{\sigma}{\sigma_0}\right)^n \qquad \text{or} \qquad \dot{\epsilon} = B\sigma^n$$

where $\dot{\epsilon}$ is the axial strain rate, σ is the axial stress, $\dot{\epsilon}_0$ and σ_0 are proof strain rate and proof stress, respectively, and B and n are creeping constants. Typically, n is about 3 (Morgenstern *et al.* 1980) for ice at low stresses and icy silts (McRoberts *et al.* 1978). In search for the dependence of n and B on temperature, Morgenstern *et al.* (1980) found from analyses of available creep data that ice behaves more as a linearly viscous material at temperatures close to 0°C. The constant B is found to be temperature and material dependent.

The motivation for studying the behaviour of a pipeline (beam) on an elastoplastic foundation is that, for ice, n is found to be within the range of 3 to 4 and this is sufficiently large so as to be analogous to a rigid–plastic material ($n \to \infty$). Of course, the material behaviour is linearly viscous when $n = 1$ in a Norton-type relationship. The former aspect has been studied (Rajani and Morgenstern, in preparation), and here we confine ourselves to n between the bounds indicated above. In this paper we present the solution of a beam on a creeping foundation, and an approximate three-dimensional (3D) solution is proposed.

Since the development of the solution is of a general nature in that it can be readily adapted to the analysis of a pipeline or a pile free at the head, we shall refer to either structure as a beam and the surrounding medium as the foundation.

Review of previous work

Flügge (1975) presented the solution of a finite beam on a linear viscous foundation, i.e., $n = 1$. Flügge's solutions are based on the correspondence principle, which states that when a viscoelastic system is subjected to a constant load or displacement, then displacements or stresses depend on time and are calculated in the same manner as those in an elastic system, except that the elastic material properties are replaced by viscoelastic parameters. As indicated by Flügge (1975), the conditions of equilibrium, kinematics, and constitutive relationships must be satisfied for the correspondence principle to be applicable for the analysis of viscoelastic systems. It is important to point out that this procedure, in which the time variable is separated in the analysis, is also often referred to as Hoff's elastic analogy (1954). The application of the correspondence principle is not limited to linear elastic systems as shown by Hoff (1954). A direct consequence of the correspondence principle or Hoff's elastic analogy is that the distribution of stresses in space within the system remains constant. This phenomenon is often referred to as stationary creep, and it needs to be contrasted with secondary creep where creep occurs with a constant rate of strain.

In an indeterminate structure where the material behaves according to the secondary creep law (eq. [1]), a redistribution of stresses occurs during the transient phase and the structure behaves as though the material were subjected to primary creep. Hence, to distinguish this phenomenon from primary creep, which is a material property, it is often referred to as statical creep. In the present analysis, we are dealing with an indeterminate structure (i.e., beam in a creeping formation), and we will observe statical creep, which should not be confused with primary creep.

Most of the developments for the solution of a beam in a creeping medium have taken place with particular reference to laterally loaded piles in permafrost. Furthermore, all these developments consider the foundation to be of the Winkler type, which makes the problem more amenable to a simple solution. Early solutions proposed by Ladanyi (1973) and Rowley *et al.* (1973) were essentially along the same lines as the nonlinear analysis of laterally loaded piles in unfrozen ground, i.e., nonlinear p–y representation of the frozen ground, where p is intensity of pressure. Nixon (1984) dealt with a short rigid pile as well as a flexible pile embedded in frozen soil that follows the secondary creep law (eq. [1]). In the case of a flexible pile, Nixon (1984) established a differential equation treating the foundation as a creeping Winkler foundation and solved it numerically using the finite difference technique. More recently, Foriero and Ladanyi (1990) have proposed a solution where the lateral reaction due to creep is represented by Maxwell springs and the creep displacements of the surrounding medium are evaluated using finite elements. Almost always, the effectiveness of the different methods have been demonstrated by comparing the predictions with pile load test carried out by Rowley *et al.* (1973, 1975).

After looking in detail at the different solutions and strategies it is evident that comparing the solutions on a case by case basis does not permit us to gain insight and hence develop an understanding of the role of the different parameters. A more comprehensive analytical framework is desirable.

Consequently, in the present paper, we attempt to obtain upper and lower bound analytical approximate solutions that enhance the understanding of the behaviour of a laterally loaded beam in a creeping medium. In nonlinear finite element analysis of lateral loads in unfrozen soil, discrete springs are often used in which the spring characteristics that are assigned correspond to the nonlinear behaviour of the foundation response. Using simple energy concepts, we develop simple relations for defining spring characteristics when the foundation follows the material law as described by [1]. This permits the use of conventional finite element programs for analyzing these types of problems. In a previous paper (Rajani and Morgenstern, in preparation), we presented the solution to the problem of a beam in an elastoplastic foundation, i.e., the limiting solution when $n \to \infty$.

TABLE 1. Indentation factors

Feature	Indentation factor I_n	Reference
Cavity expansion strip footing solution	$I_n = \dfrac{n}{\sqrt{3}}\left(\dfrac{8}{\pi\sqrt{3}}\right)^{1/n}$	Nixon (1978); Ladanyi (1983)
Flat indenter on semi-infinite half-space	$I_n = \dfrac{\phi}{(\phi\psi)^{1/2}}$, where $\phi = \dfrac{\pi + 2}{\sqrt{3}}$, and $\psi = 0.445$	Ponter *et al.* (1983)
Long cylinder streamline solution (plane strain)	$I_n = \dfrac{2\pi}{\sqrt{3}}\left(\dfrac{8}{\sqrt{3}}\right)^{1/n} \dfrac{n^2}{(n + 1)(n + 3)}$	Foriero and Ladanyi (1989)

Beam embedded in a creeping medium

For the present analysis we assume that the beam is buried in a homogeneous and isotropic, elastic and nonlinear viscous medium and that, when subjected to uplift, the beam deforms with a double curvature. We recognize that, in fact, for shallow pipelines this may not be totally valid. The creep behaviour of the medium is represented by a Norton-type relation (eq. [1]). If the elastic subgrade modulus is represented by k_s, then the foundation stiffness k_s' is given by $k_s' = bk_s$, where b is the beam width (pile or pipeline diameter). Nixon (1978) has related the displacement rate ($\dot{w}$) of a long cylinder to the stress on the loaded area, and this is given by

$$[2a] \quad \dot{w} = \left(\frac{I_f Bb}{2}\right)p^n$$

where p is the intensity of pressure on the loaded area, and I_f is the influence factor dependent on n and the geometry of the loaded beam. Since the beam width is b, then the reaction per unit length q is given by

$$[2b] \quad \dot{w} = \frac{\left(\frac{I_f Bb}{2}\right)}{(b)^n}q^n = B'q^n$$

and $B' = (I_f Bb/2)/(b)^n$ is the creep compliance coefficient for the foundation. The above relation can be rewritten in the general form

$$[3] \quad F_z = I_n b\sigma_o\left(\frac{\frac{\dot{w}}{b}}{\dot{\epsilon}_o}\right)^{1/n}$$

where $F_z(= pb)$ is the resistance per unit length offered by the surrounding medium, and $I_n(=(2/I_f)^{1/n})$ is the indentation factor. Indentation factors for a von Mises material as determined by Nixon (1978), Ladanyi (1983), Ponter *et al.* (1983), and Foriero and Ladanyi (1989) are shown in Table 1. We note from [3] that as $n \to \infty$ the indentation factor becomes Prandtl's bearing capacity factor N_c. While indentation factors for a flat indenter and a circular disk (plane strain) as determined by expressions in Table 1 approach Prandtl's limiting value, the cavity expansion solution for a strip footing is unbounded. The slight difference in the solution for the long cylinder and the flat indenter studied by Ponter *et al.* (1983) is due to the shape of the two indenters. The variation of the indentation factor with creep coefficient n is shown in Fig. 2 and demonstrates that it is preferable to use the indentation factor as proposed by Foriero and Ladanyi (1989).

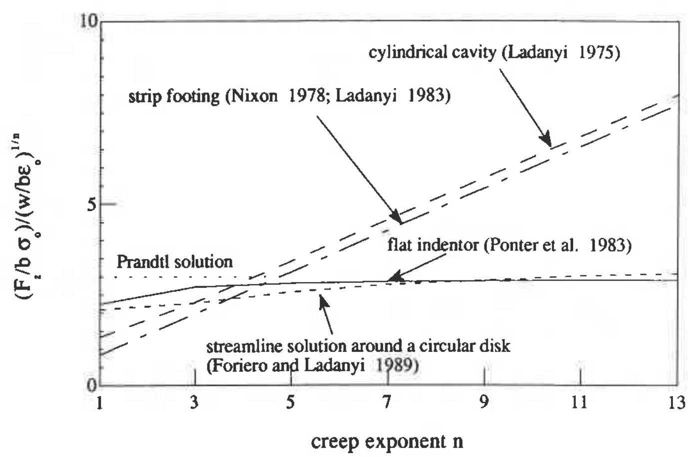

FIG. 2. Indentation factors for a semi-infinite creeping medium (von Mises material).

A consequence of the double curvature mentioned earlier is that the transition point O (Fig. 3) is a point of inflexion implying a stress boundary condition of zero moment. On the other hand, if we choose to look at the problem as that of a pile subjected to a lateral load P, then the equivalent problem of a pipeline subjected to frost heave would be given by a prescribed displacement of $w_o = P\beta/k_s'$, where $\beta^4 = k_s'/4EI$ and where E is the elastic modulus and I is the moment of inertia for the pipeline. We also note that the resulting problem is statically indeterminate.

The variational approach to determine the governing differential equation for equilibrium is adopted to establish bounded solutions. We will invoke the stationary condition for the total potential (Π), i.e., $\delta\Pi = 0$. The problem can be conveniently separated into two time frames. On initial application of the load P, i.e., at $t = 0$, there will be an immediate elastic response (w_e), and this response can be determined by the usual beam on elastic foundation type solutions. Subsequent creep response (w_c) will depend on the interaction of the beam and the creeping characteristics of the foundation material. The total (accumulated) response can be estimated by the application of superposition of states which can be expressed as

$$[4] \quad w(t) \approx w_e(0) + \int_0^t \dot{w}_c(t)\,\mathrm{d}t$$

The above approximation defines the superposition of an elastic response, determined as if there were no creep, and

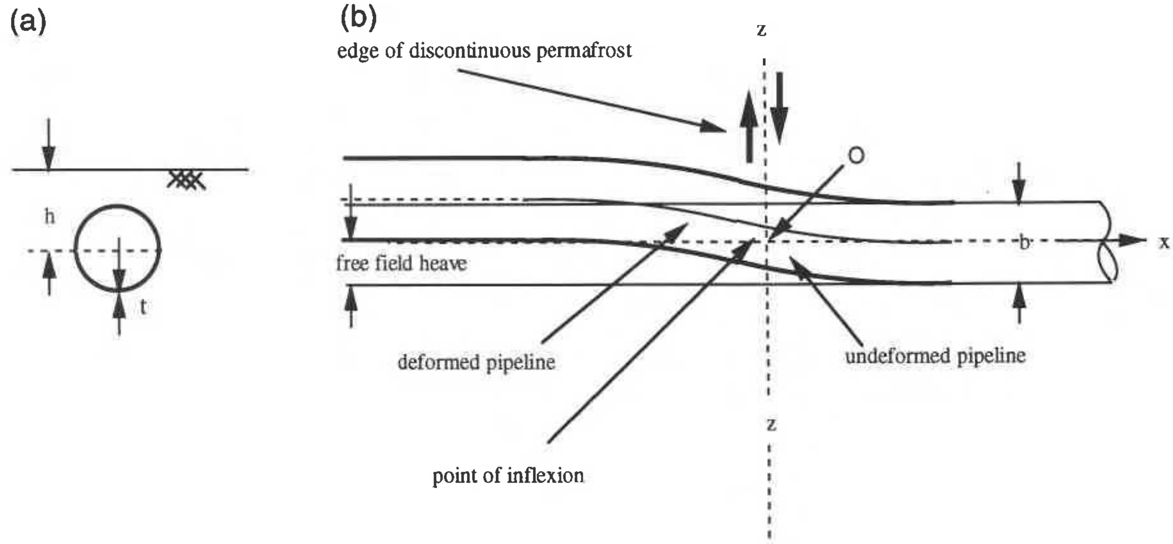

FIG. 3. Beam embedded in a creeping medium. (*a*) Cross section. (*b*) Longitudinal section.

a pure creep response, determined as if there were no elastic response.

The elastic response can be obtained readily from Hetenyi (1974). The total potential (Π) in rate form (Boyle and Spence 1983) for the creep steady-state resposne of the semi-infinite beam under end load P, where the beam follows the small-displacement elastic Euler–Bernoulli relation and the foundation obeys the Norton-type creep law, is given by

$$[5] \quad \Pi = \frac{1}{2}\int_0^\infty \dot{w}_c'' M \,\mathrm{d}x + \frac{n}{(1+n)}\int_0^\infty q\dot{w}_c \,\mathrm{d}x - P\dot{w}_c(0)$$

where $\dot{w}_c$ is the creep displacement rate at the point of application of the load P, and M is the bending moment. Upon substituting for the moment M according to the Euler–Bernoulli theory, i.e., $-M = EIw''$, and the medium reaction q from [2*b*], we get after integration by parts

$$[6a] \quad EI\frac{\partial^4 w_c}{\partial x^4} + \left(\frac{\dot{w}_c}{B'}\right)^{1/n} = 0$$

$$[6b] \quad EIw_c''|_{x=0} = 0$$

$$[6c] \quad [EIw_c''' + P]|_{x=0} = 0$$

The governing differential equation is given by [6*a*], and the natural boundary conditions are the relations in [6*b*] and [6*c*]. We note that these natural boundary conditions correspond to the physical conditions of moment and shear equilibrium at $x = 0$. The differential equation expressed in [6*a*] is similar to that obtained by Nixon (1984). Analytical solutions for [6*a*] are very difficult to obtain for $n \neq 1$, and hence we have to resort to either approximate or numerical techniques. However, for a linear viscous foundation, i.e., $n = 1$, an exact solution for the equation can be found. There exists an interest in a solution for this particular case from a practical point of view. Morgenstern *et al.* (1980) have shown that frozen soil behaviour can be idealized as a linear viscous material when the temperatures are near freezing (0°C). Moreover, the solution for a linear viscous foundation is readily comparable to solutions obtained by other approximate methods and provides insight into the general behaviour that is often obscured by other solution techniques.

Linear viscous foundation, i.e., $n = 1$

We obtain the following solution upon integrating [6*a*] with respect to time and applying the boundary conditions expressed by [6*b*] and [6*c*]:

$$[7] \quad w_c = 2P\beta_c N' t e^{-\beta_c x}\cos\beta_c x$$

where $\beta_c^4 = \dfrac{1}{4EIB't}$. Hence the total response can be determined as stated in [4] and is given by

$$[8] \quad w = (2P\beta/k_s')[e^{-\beta x}\cos\beta x + (k_s'B't)^{3/4}e^{-\beta_c x}\cos\beta_c x]$$

The end displacement at the point of application of the load is

$$[9] \quad w = \frac{2P\beta}{k_s'}[1 + (k_s'B't)^{3/4}]$$

In the case of a beam on an elastic foundation, the characteristic length β represents the relative elastic stiffness of the beam and foundation. The solution given by [7] for the creeping foundation results in an equivalent characteristic length that is an inverse function of time t and the creep parameter B'. This solution clearly demonstrates why the technique of solving a beam on a creeping foundation using a time-dependent k-modulus has been successful.

An approximate upper bound solution for a beam on a linear viscous foundation according to the Rayleigh–Ritz method described later is

$$[10] \quad w = \frac{2P\beta}{k_s'}[8 - 7e^{-k_s'B't/6}]$$

The reason why a separate solution has to be sought when $n = 1$ will be discussed when dealing with the Rayleigh–Ritz method of analysis. A lower bound solution for a beam on a linear viscous foundation obtained based on Martin's inequality described later is given by

$$[11] \quad w = \frac{2P\beta}{k_s'}\left(1 + \frac{3k_s'B't}{7}\right)$$

The total response as expressed by [8] clearly shows that the initial static response sets up a stationary stress wave along the beam–foundation system, and the subsequent creep response sets up another stress wave that is of similar

TABLE 2. J integrals for particular values of n

Creep exponent n	J_1 integral	J_2 integral
1	0.375000	0.375000
3	0.538126	0.209375
5	0.595879	0.146635
7	0.625020	0.113014

shape to that of the elastic response. Furthermore, the creep stress response is modified by the so-called reduced k-modulus and propagates with time towards the semi-infinite end. Though this argument has been demonstrated for linear viscous foundations, similar effects will be present for foundations with $n \neq 1$. These findings are in complete accordance qualitatively with field experimental results of a buried chilled pipeline facility at Caen, France, as reported by Dallimore and Crawford (1984).

Creeping foundation, i.e., n $\neq$ *1*

As stated before, for a creeping foundation, direct solution of the differential equations becomes almost impossible for $n \neq 1$. Consequently, we have to resort to approximate techniques where we can obtain reasonable estimates and bounds. In this paper, we will approach the problem using two methods that guarantee upper and lower bounds. The two methods are essentially based on the minimization of the total potential Π; the first approach is the Rayleigh–Ritz method, and the second approach is based on the application of Martin's inequality.

Upper bound: Rayleigh–Ritz

The theorem of minimum potential energy dissipation states that, for the particular case of a beam on a creeping foundation, amongst all kinematically admissible curvatures (κ^k) and displacement rates ($\dot{w}^k$), the actual curvatures and displacement rates minimize the functional Π of [5]:

$$[12] \quad \Pi(\kappa^k, \dot{w}^k) = \int_0^\infty \left[EI\kappa^k\dot{\kappa}^k + \frac{n}{(1+n)}\left(\frac{\dot{w}^k}{B'}\right)^{1/n} \dot{w}\right] \mathrm{d}x$$

Since κ^k, $\dot{w}^k$ must be kinematically admissible, they must satisfy the natural boundary conditions obtained in [6]. An obvious choice for $\dot{w}^k$ that satisfies the required boundary conditions is that which corresponds to the spatial elastic solution. Hence, we assume for $\dot{w}^k$

$$[13] \quad w(x, t) = u(t) w_e(x)$$

In the above assumed solution, $u(t)$ is the function that depends on time. We also note that this assumed solution is also in accordance with the correspondence principle or Hoff's (1954) elastic analogy. Substituting [13] in [12] and carrying out the minimization with respect to displacement rate $\dot{u}$, we obtain

$$[14] \quad u(t) = \frac{2P\beta}{k_s'}\left\{8 - 7\,[1 + rk_s'B'\,(P\beta)^{n-1}t]^{1/(1-n)}\right\}$$

where

$$r = \frac{0.875^{(n-1)}\,(n-1)}{16J_1^n}$$

$$J_1 = \int_0^\infty (e^{-z}\cos z)^{1+1/n}\,\mathrm{d}z$$

Equation [14] identifies the dimensionless time as $rk_s'B'\,(P\beta)^{n-1}t$, and we shall see that it appears recurrently in the rest of the analysis. Once again, the static response has been factored out to isolate the displacement amplification due to creep alone. The J_1 integral can be evaluated for particular values of n, and sample values are given in Table 2. The J_2 integral given in Table 2 is defined after [24].

Equation [14] breaks down for the particular case when $n = 1$, and it is necessary to obtain a particular solution (eq. [9]), but the procedure remains the same. It is to be expected that the solution will deteriorate with increasing values of n, as equilibrium will be steadily violated. The Rayleigh–Ritz procedure together with a series solution as proposed by Heteyni (1974) for beams on elastic foundations could lead to an improvement in the accuracy of the solution. However, to do so analytically would be somewhat involved algebraically. An added advantage of this modified procedure lies in cases of beams of variable cross section EI or in which a nonlinear moment curvature relation exists.

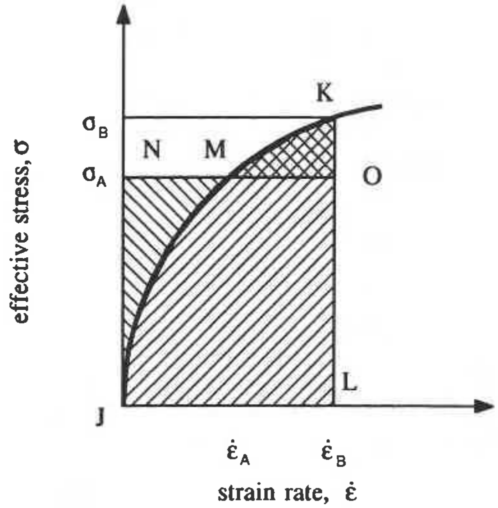

FIG. 4. Geometrical interpretation of Martin's inequality.

Lower bound: Martin's inequality

In general terms, Martin's inequality (Boyle and Spence 1983) for the power creep law can be stated as

$$[15] \quad \frac{n}{n+1}\int_V \sigma_B\dot{\epsilon}_B\,\mathrm{d}V + \frac{1}{n+1}\int_V \sigma_A\dot{\epsilon}_A\,\mathrm{d}V \geq \int_V \sigma_A\dot{\epsilon}_B\,\mathrm{d}V$$

where we identify σ_B, $\dot{\epsilon}_B$ with the actual solution σ, $\dot{\epsilon}$, and σ_A, $\dot{\epsilon}_A$ with statically admissible surface tractions along the beam–foundation interface. Martin's inequality derivation is based on a postulate for material stability formulated by Drucker (1951) which in turn ensures that the constitutive relation is monotonic, i.e., the increase in stress causes an increase in strain rate. A geometrical interpretation of [15] can be deduced from Fig. 4. The first and second terms of [12] correspond to strain and complementary energy dissipations in the areas represented by the polygons JMKOL and JNMOL, respectively. It then becomes obvious why the inequality holds true for any monotonic functional relation. In our particular beam–foundation system, the beam is assumed to behave entirely elastically, and only the foundation is composed of the creeping material. Rewriting [*15*] in terms of the moments and curvature rates and taking into account the two different components of the system, we have

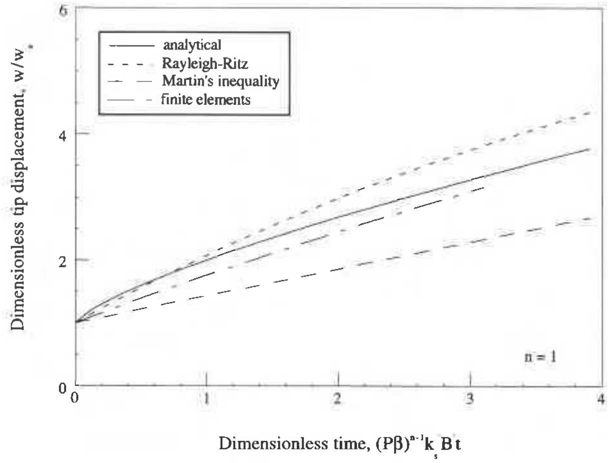

FIG. 5. Comparison of different methods of determining tip displacement with time for a semi-infinite beam on a linear viscous creeping foundation.

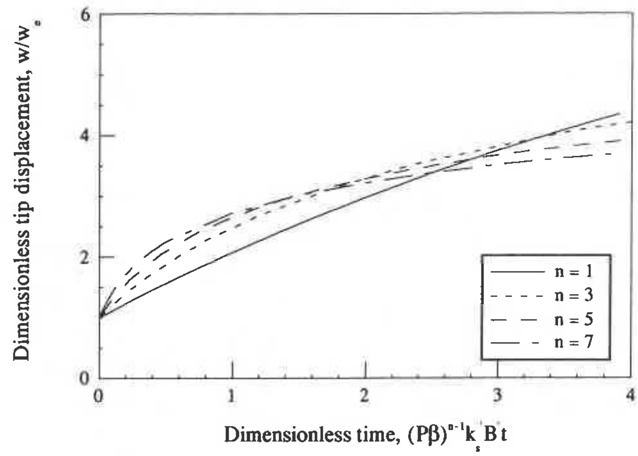

FIG. 6. Tip displacement with time for a semi-infinite beam on a nonlinear creeping foundation using the Rayleigh–Ritz method.

$$[16]\quad \frac{1}{2}\int_0^\infty \dot{w}_B'' M_B\,dx + \frac{1}{2}\int_0^\infty \dot{w}_A'' M_A\,dx + \frac{1}{(n+1)}\int_0^\infty q_A \dot{w}_A\,dx + \frac{n}{(n+1)}\int_0^\infty q_B \dot{w}_B\,dx \geq \int_0^\infty \dot{w}_B''\, M_A\,dx + \int_0^\infty \dot{w}_B q_A\,dx$$

where $\dot{w}$, $\dot{w}''$, M, and q refer to transverse displacement, curvature, moment, and foundation reaction, respectively. Upon substituting for $\dot{w}$ using the creep material relation defined by [2] and M by the Euler–Bernoulli moment curvature relation and if we let the B state correspond to the true solution (without subscript), we obtain

$$[17]\quad \frac{1}{2}\int_0^\infty \dot{w}'' M\,dx + \frac{1}{2}\int_0^\infty \frac{\dot{M}_A M_A}{EI}\,dx + \frac{B'}{(n+1)}\int_0^\infty q_A^{n+1}\,dx + \frac{nB'}{(n+1)}\int_0^\infty q^{n+1}\,dx \geq \int_0^\infty \dot{w}'' M_A\,dx + \int_0^\infty \dot{w} q_A\,dx$$

Integrating the first term on the right- and left-hand sides of inequality [16] by parts twice and after rearranging terms, we obtain

$$[18]\quad \frac{1}{2}\dot{w}'M\Big|_0^\infty - \frac{1}{2}\dot{w}\frac{dM}{dx}\Big|_0^\infty + \dot{w}\frac{dM_A}{dx}\Big|_0^\infty - \dot{w}'M_A\Big|_0^\infty + \int_0^\infty\left(-\frac{d^2M_A}{dx^2} + \frac{1}{2}\frac{d^2M}{dx^2} - q_A + \frac{n}{(n+1)}q\right)\dot{w}\,dx \geq -\frac{1}{2}\int_0^\infty \frac{\dot{M}_A M_A}{EI}\,dx - \frac{B'}{(n+1)}\int_0^\infty q_A^{n+1}\,dx$$

The essential boundary conditions are given by the first two terms on the left-hand side of the inequality, and the natural boundary conditions are given by the following two terms. We have not imposed any conditions on M_A or q_A except that they be statically admissible. One must find adequate functions of M_A and q_A that satisfy the integral on the left-hand side of the inequality (eq. [18]). We then select a function for q and M on the basis of elastic beam foundation systems which will ensure that M_A and q_A satisfy the stated boundary conditions. Two governing differential equations can be set from the integral on the left-hand side of the inequality. They are

$$[19a]\quad \frac{d^2M_A}{dx^2} + \frac{n}{(n+1)}q = 0$$

$$[19b]\quad \frac{1}{2}\frac{d^2M}{dx^2} - q_A = 0$$

The true static elastic foundation reaction set up is given by

$$[20]\quad q = -4P\beta\, e^{-\beta x}\cos\beta x$$

Substituting in [19a] and integrating once we find that the shear at $x = 0$ is given by

$$[21]\quad S = -\frac{2nP}{(1+n)}$$

We note that the natural boundary condition is satisfied when $n = 1$ and deteriorates on increasing n value. Hence, we should expect that the solution will be poor for high values of n. A further integration of [19a] leads to an expression for M_A:

$$[22]\quad M_A = \frac{2n}{(n+1)}\frac{P}{\beta}\, e^{-\beta x}\sin\beta x$$

The expression for M_A satisfies the natural boundary conditions stated earlier. Recognizing that $d^2M/dx^2 - q = 0$ from beam theory, we can subsequently obtain an expression for q_A:

$$[23]\quad q_A = -2P\beta\, e^{-\beta x}\cos\beta x$$

Following substitution of expressions for M_A and q_A in the right-hand side of the inequality and upon application of the correspondence principle as stated in [12], we obtain after integration

$$[24]\quad u(t) \geq \frac{(-1)^{n+1}\,2^{n+1}\,(n+1)}{(n^2+4n+2)}\, J_2 B' k_s' t (P\beta)^{n-1}$$

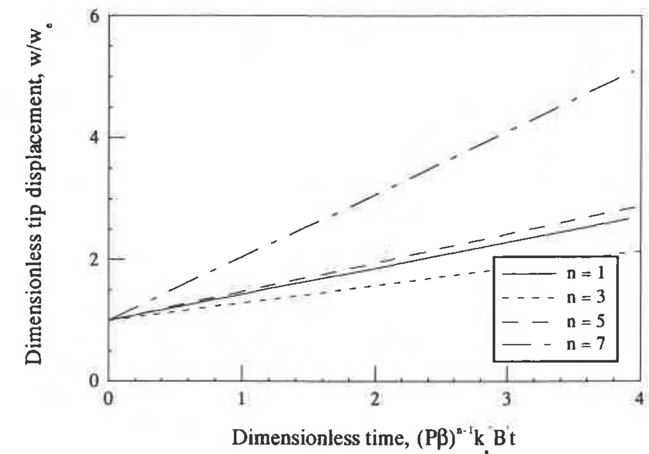

FIG. 7. Tip displacement with time for a semi-infinite beam on a nonlinear creeping foundation using Martin's inequality.

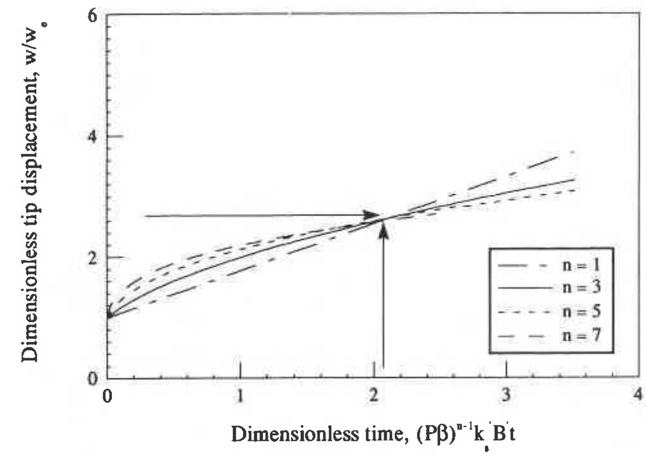

FIG. 8. Tip displacement with time for a semi-infinite beam on a nonlinear creeping foundation using the finite element method.

where $J_2 = \int_0^\infty (e^{-z}\cos z)^{n+1}\,dz$. J_2 has been tabulated for different values of n in Table 2. Once again we note the recurrent dimensionless time parameter in [24]. A much improved solution is obtained if the corrected shear as stated in [21] is used. The corresponding solution is given by

$$[25] \quad u(t) \geq \frac{(-1)^{n+1}2^{n+1}(n+1)}{(4n^2+4n-1)} J_2 B' k_s' t (P\beta)^{n-1}$$

Nonetheless, it should be emphasized that the solution should deteriorate for large values of n but, as we shall see later for the range of values of n that we are concerned with, a good approximate solution is obtained.

Finite elements

Although the above solutions provide insight into the behaviour and understanding of the system, they do limit the analysis to ideal situations, i.e., homogeneous medium. In the last three decades the finite element method has proved immensely useful in solving problems previously intractable by analytical or approximate means. The solution of beams on nonlinear foundations is now routinely carried out using available finite element codes. One code that has incorporated within it a truss element with the Norton–Bailey creep law is the finite element code ADINA (ADINA R&D, Inc. 1987). However, the characterization (i.e., spacing of springs, cross-sectional area, length, etc.) of the truss springs has to be done with care so that equivalence is maintained between the beam on a continuous creeping foundation and the beam on a discretized foundation.

It is natural to expect that, as spacing of the discretized truss springs becomes small, the resulting approximation will be accurate. In a practical situation we wish to get away with as few truss springs as possible. Boresi *et al.* (1978) have shown that spacing s of truss springs can be estimated by

$$[26] \quad s = \frac{\pi}{4\beta}$$

We note that, although the above criterion is developed for a beam on an elastic foundation, we observe from the development of the solution for a beam on a linear viscous foundation that the characteristic length β_c is essentially the same as β except that the effective foundation modulus is time dependent (inverse relation). Consequently, for the steady-state solution, if necessary, the spacing(s) could be steadily increased with time, and thus the spacing of the springs guided by [26] is more than adequate for the complete analysis.

The material characteristic of the truss spring has to be adequately represented for the elastic response and the creep response. Hence, if we arbitrarily fix the length of the springs to say, L, then conflicting definitions of cross-sectional areas arise in attempting to satisfy both responses. A plausible way to remedy this situation is to first satisfy the requirements of creep response and then redefine the elastic modulus of the spring. In order that the spring represents adequately the creep response, the rate of work done by the spring should be equal to that done by the continuous creeping medium, i.e.,

$$[27] \quad (Q\dot{w})_{\text{spring}} = (bqs\dot{w})_{\text{creeping foundation}}$$

where Q and q are the axial and foundation reactions in the spring and continuous medium, respectively. If the material properties of both the spring and the continuous foundation are defined as in [1] and [2], then on substitution in [27] we obtain the cross-sectional area of the spring A_{spring} as

$$[28] \quad A_{\text{spring}} = bs\left(\frac{2L}{bl_f}\right)^{1/n}$$

Consequently, the elastic modulus of the springs E_{spring} has to be

$$[29] \quad E_{\text{spring}} = \frac{sLk_s'}{A_{\text{spring}}}$$

The above characterization of creep springs is independent of the particular geometry and loading of the beam–foundation system. A pertinent question that may be posed is how long should the beam–foundation system be to simulate a semi-infinite condition. Den Hartog (1952) has provided an elegant argument for this problem which essentially concludes that if $\beta X \geq 4$ (where X is the length of beam–foundation system) then the beam–foundation system can be treated as a semi-infinite beam. Since we are dealing with a nonlinear viscous creeping foundation, the corresponding creep characteristic length should be used, i.e.,

$$[30] \quad X^{\text{creep}}_{\text{semi-infinite}} \geq 4\sqrt[4]{4EIB'\,t(P\beta)^{n-1}}$$

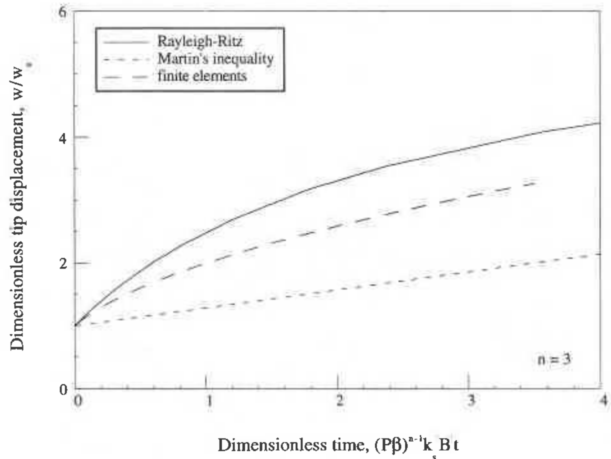

FIG. 9. Comparison of different methods of determining tip displacement with time for a semi-infinite beam on a nonlinear (n = 3) creeping foundation.

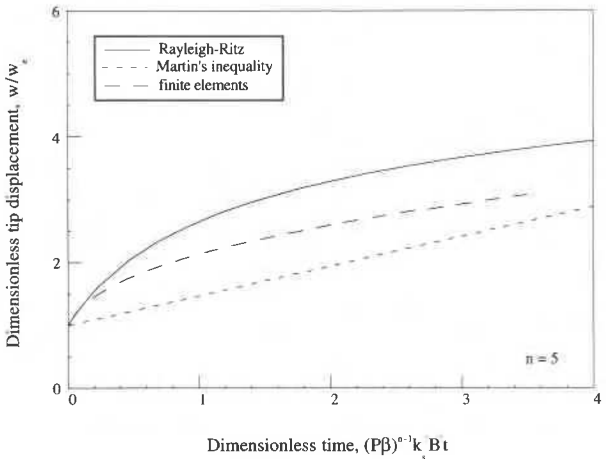

FIG. 10. Comparison of different methods of determining tip displacement with time for a semi-infinite beam on a nonlinear (n = 5) creeping foundation.

We note from the above that length ($X^{\text{creep}}_{\text{semi-infinite}}$) of the beam–foundation system extends with time and should be added to the length obtained for elastic response. Hence, provision should be made at the outset for the time span of the analysis so that adequate length of the beam–foundation system is always ensured.

Various techniques have been proposed for carrying out time integration for solving creep problems. An implicit time integration, α-method, was used in the analysis, as explained by Bathe (1982). When $\alpha = 0$, the method reduces to the Euler forward method, and when $\alpha > 0$ it is the implicit method. One particular advantage of the implicit time integration scheme is that the method is unconditionally stable for $\alpha > ½$ for any time step size, though it may not necessarily converge to the correct solution.

Comparison between the different solutions

As indicated before through the approximate analyses, the nondimensional time factor $\bar{t}$ $(=k'_sB'(P\beta)^{n-1}t)$ has been identified and conveniently provides a basis for adequate comparisons between the different methods. It has become customary to evaluate the response for the types of

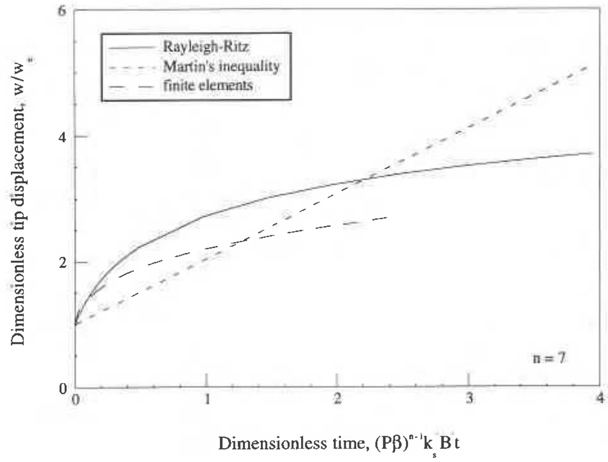

FIG. 11. Comparison of different methods of determining tip displacement with time for a semi-infinite beam on a nonlinear (n = 7) creeping foundation.

problems considered in this paper to either measures of the displacement or displacement rates, and here we use the former measure, since both features can be readily appreciated from a single plot. Figure 5 shows the comparison between all the methods discussed in this paper for a linear viscous foundation creeping material. The approximate upper and lower bounds provide reasonably good estimates if we compare them with the available exact solution. The finite element solution is comparably close to the exact solution and could be improved with finer discretization and using smaller time steps in the time integration procedure. Figures 6–8 show displacement–time histories for Rayleigh–Ritz, Martin's inequality, and finite element solutions, respectivley, for varying values of n. The solutions as obtained using the Rayleigh–Ritz and the finite element methods clearly show the role of statical creep indicated earlier. Also, these methods reveal an interesting phenomenon that there exists a particular nondimensional time when the response is independent of the creep exponent n. As expected this effect is masked when a lower bound solution is obtained using Martin's inequality. If we compare the steady-state responses in terms of displacement rates, then the comparison is good. Similarly, Figs. 9–11 show the same previous responses except that each solution is compared with each other for particular n values of 3, 5, and 7, respectively. The solution obtained using Martin's inequality is insensitive to $n < 5$ but diverges for $n > 5$ and thus no longer represents a lower bound. This is to be expected as discussed earlier, since one of the natural boundary conditions (end shear) is increasingly violated for increasing n values. If, on the other hand, the response is measured in terms of steady state (displacement rates), then all the approximate methods compare well.

Conclusions

The various solutions have highlighted the role of the different parameters in the total response of a beam embedded in a creeping foundation. Simplified analytical upper and lower bounds for a beam in a creeping foundation subjected to an end load have been developed. A simple finite element modelling procedure has been outlined which facilitates the general solution for this type of problems if a truss spring with Norton's creep law is available in a general finite element code. The finite element analysis confirms the bounds established using variational principles. Although the upper bound established using the Rayleigh–Ritz approach exhibits all the basic characteristics shown to exist by the finite element method, the lower bound estimates are poor because of the inability of the selected functions to satisfy the natural boundary conditions for all values of the creep exponent n. Nonetheless, the analytical solutions developed here remain upper and lower bounds as long as $n \leq 5$. All the results have been obtained in nondimensional form, which permits rapid evaluation for design purposes.

B. Rajani and M. Morgenstern (in preparation) have demonstrated how the bearing capacity factor N_c can be adjusted to account for finite shallow burial. Through [3] we have shown how the indentation factor I_n tends towards the Prandtl bearing capacity factor N_c for large values of n. Consequently, the indentation factor I_n can be similarly adjusted to account for shallow burial of pipelines.

The solutions permit the rapid evaluation of the amplification of the elastic response of a beam embedded in a creeping foundation. To obtain an accurate creep response it is equally important to obtain a proper estimate of the elastic response. The application of the above solutions for pipelines and laterally loaded piles will be treated in a subsequent publication. It is pertinent to indicate that the approximate solutions presented in this paper cannot be directly applied to the case of pipelines subjected to steady upwards movement, i.e., frost heave. Our purpose in developing approximate analytical solutions was to confirm the validity of the proposed finite element scheme. Finite element analysis is a flexible tool that permits the specification of arbitrary boundary conditions, loading patterns, and geometries. In fact, the finite element solution for a pipeline subjected to a steady displacement is only valid as long as the end displacement applied is slow enough so that the load increment between a specific time interval is nearly constant.

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List of symbols

A_{spring}	cross-sectional area for discretized foundation spring
b	beam width, pipeline or pile diameter
B	creep proportionality constant
B'	creep foundation compliance for plane strain
c	cohesion
E	beam elastic modulus
E_s	soil elastic modulus
E_{spring}	elastic modulus for discretized foundation spring
h	embedment depth
I	beam moment of inertia
I_f	creep influence factor
I_n	creep indentation factor
J_1, J_2	integrals dependent on creep exponent n
k_s'	foundation stiffness
k_s	foundation subgrade modulus
L	length of discretized foundation spring
M	bending moment
p	reaction intensity pressure
P	nondimensional load parameter
n	creep exponent in Norton relation
N_c	Prandtl bearing capacity factor
q	reaction per unit length
Q	axial force in discretized foundation spring
s	discrete spring spacing
S	shear
$\bar{t}$	nondimensional time parameter
u, $\dot{u}$	displacement and displacement rates time dependent functions
V	volume
w_c, $\dot{w}_c$	creep displacement and displacement rates
w_e	elastic displacement
$\dot{w}$	accumulated transverse displacement
w	transverse displacement in the z-direction
$\overline{w}$	nondimensional displacement parameter
x	longitudinal coordinate axis
$X^{\text{creep}}_{\text{semi-infinite}}$	additional length of beam–foundation system for discretization during creep
z	axis normal to x-axis
α	implicit integration parameter
β, β_c	elastic and linear viscous characteristic lengths
$\dot{\epsilon}_o$	proof strain rate
γ	soil weight density
Π	total potential
ν_s	soil Poisson's ratio
κ	curvature
σ, σ_o	stress and proof stress
σ_y	yield stress of surrounding medium