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A NOTE ON THE APPLICATION OF FORMAC TO A DIFFRACTION PROBLEM

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ABSTRACT

The symbolic programming language FORMAC is used to find the coefficients in the power series expansion of the aperture field and transmission coefficient for a small circular aperture in a rigid plane screen. Terms including the fourteenth and sixteenth powers of the ratio 2π radius/wavelength are obtained for the transmission coefficient and the aperture field, respectively.

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Some years ago, recurrence relations were developed for the coefficients in the expansion of the scalar field in a small circular aperture in a hard screen (Hurd, 1961). If $h(x)$ is the aperture distribution and $\alpha = 2\pi\cdot\text{radius}/\lambda$, it was found that

$$h(x) = \sum_{n=0}^{\infty} \alpha^n h_n(x) \tag{1}$$

with

$$h_n(x) = \sum_{m=0}^{[n/2]} \zeta_{2m}^n x^{2m} \quad , \tag{2}$$

x being the radial co-ordinate normalized to the radius of the aperture. The coefficients ζ_{2m}^n are the solutions of a pair of equations:

$$\zeta_{2m}^n = 2\beta \sum_{r=m}^{[(n-1)/2]} \binom{2r}{2m} \frac{1}{(2r+1)!} \rho_{2r-2m+1}^{n-2r-1} + \delta_{2m,n}/n\,!, \tag{3a}$$

$$\rho_t^s = \sum_{m=0}^{[s/2]} \zeta_{2m}^s / (t+2m), \tag{3b}$$

where $\beta = -i/\pi$, $\delta_{2m,n}$ is the Kronecker delta, and the ρ_t^s are certain auxiliary functions.

It was also shown that the transmission coefficient τ, given by

$$\tau = \sum_{n=0}^{\infty} \alpha^{2n} \tau_{2n} , \tag{4}$$

could be calculated from

$$\tau_{2n} = -4\beta \sum_{m=0}^{n} [(2m)!]^{-1} \rho_{2m+1}^{2n-2m+1} \quad . \tag{5}$$

Using the initial condition $\zeta_0^0 = 1$, it is evident that ρ_t^s and ζ_{2m}^n of all orders can be calculated from (3). It is also clear that the amount of work increases rapidly with n. In the reference, terms including ζ_{2m}^{12}, τ_{10} were found, but this involved several weeks' work.

The calculation has now been repeated using FORMAC – a programming language which manipulates algebraic symbols (Tobey *et al*, 1967). While its usefulness as an aid to isolated algebraic calculations appears to be somewhat limited, FORMAC does seem to be very helpful for calculations of a repetitive kind, such as occur above.

Before programming, the coefficients ρ^s_t were first eliminated from (3), resulting in a double series for the calculation of the ζ^n_{2m} . The double series was then solved in terms of the parameter β by FORMAC.

A copy of the program is shown in Table 1. In it the symbols ZETA(N,M), TAU(N), and B correspond to ζ^n_m, $\frac{\pi^2}{4}\tau_n$, and β respectively in the notation above. While no particular effort was made to keep the program short, the economy of instructions is still quite remarkable. Terms including ζ^{16}_{2m}, τ_{14} were computed and the results agree exactly with the previous calculation in the region of overlap. Table 2 shows the new values, i.e. from ζ^{13}_0 on. Also included for purposes of comparison is $\frac{\pi^2}{4}\tau_{10}$. The time of execution was 3.52 minutes.

TABLE 1

```
INPUT TO FORMAC PREPROCESSOR
 SERA: PROCEDURE OPTIONS(MAIN);
 FORMAC_OPTIONS;
 OPTSET(LINELENGTH=72);
     PUT PAGE;
/* CALCULATE ZETA */
     LET(ZETA(0,0)=1);
     DO N=1 TO 16;
     DO M=0 TO FLOOR(N/2);
     LET(M="M";N="N");
     LET(SUM=0;A=0);
     IF M=N/2 THEN DO; LET(A=1); GO TO LAST;END;
     DO I=M TO FLOOR((N-1)/2);
     DO J=0 TO FLOOR((N-2*I-1)/2);
     LET(I="I";J="J");
     LET(SUM=SUM+ZETA(N-2*I-1,2*J)/(2*I+1)/(2*(1-M+J)+1)/FAC(2*I-2*M));
     END;END;
 LAST:LET(ZETA(N,2*M)=EXPAND(2*B*SUM/FAC(2*M))+A/FAC(N));
     PRINT_OUT(ZETA(N,2*M));
     END;
/* CALCULATE TAU */
     IF N ¬= 2*FLOOR(N/2) THEN GO TO FIN;
     LET(ADD=0);
     DO P=0 TO (N-2)/2;
     DO Q=0 TO P;
     FIXEDA(P=P;Q=Q);
     LET(ADD=ADD+ZETA(2*P+1,2*Q)/(N-1+2*Q-2*P)/FAC(N-2-2*P));
     END;END;
     PUT SKIP(3);
     LET(TAU(N-2)=EXPAND(ADD/B));
     PRINT_OUT(TAU(N-2));
     ATOMIZE (TAU(N-2));
     PUT SKIP(3);
FIN: END;END SERA;
```

TABLE 2

$$TAU(10) = 1280408/13395375\, B^{2} + 161408/35721\, B^{4} + 56576/675\, B^{6} + 2048/3\, B^{8} + 2048\, B^{10} + 658136/1620840375$$

$$ZETA(13,0) = 8/79053975\, B + 21628192/8104201875\, B^{3} + 29004128/66976875\, B^{5} + 5590016/297675\, B^{7} + 228352/675\, B^{9} + 8192/3\, B^{11} + 8192\, B^{13}$$

$$ZETA(13,2) = 16/9555975\, B + 3805684/736745625\, B^{3} + 5458016/13395375\, B^{5} + 425536/42525\, B^{7} + 13312/135\, B^{9} + 1024/3\, B^{11}$$

$$ZETA(13,4) = 367/109459350\, B + 6124/3274425\, B^{3} + 608/7875\, B^{5} + 64/63\, B^{7} + 64/15\, B^{9}$$

$$ZETA(13,6) = 61/34054020\, B + 3271/14033250\, B^{3} + 3/1701\, B^{5} + 8/315\, B^{7}$$

$$ZETA(13,8) = 59/194594400\, B + 13/1122660\, B^{3} + 1/11340\, B^{5}$$

$$ZETA(13,10) = 1/64864800\, B + 1/4989600\, B^{3}$$

$$ZETA(13,12) = 1/3113510400\, B$$

$$ZETA(14,0) = 77269688/4474059715125\, B^{2} + 1586864/99235125\, B^{4} + 62065664/40186125\, B^{6} + 9385216/178605\, B^{8} + 183296/225\, B^{10} + 53248/9\, B^{12} + 16384\, B^{14}$$

$$ZETA(14,2) = 11211484/147496474125\, B^{2} + 1464824/63149625\, B^{4} + 3315584/2679075\, B^{6} + 23552/945\, B^{8} + 9728/45\, B^{10} + 2048/3\, B^{12}$$

$$ZETA(14,4) = 2267297/44695901250\, B^{2} + 141566/21049875\, B^{4} + 2888/14175\, B^{6} + 2144/945\, B^{8} + 128/15\, B^{10}$$

$$ZETA(14,6) = 98897/8939180250\, B^{2} + 14422/21049875\, B^{4} + 92/8505\, B^{6} + 16/315\, B^{8}$$

TABLE 2 (Cont'd)

$ZETA(14,8) = 8963/8756748000\ B^{2} + 1/35640\ B^{4} + 1/5670\ B^{6}$

$ZETA(14,10) = 7/166795200\ B^{2} + 1/2494800\ B^{4}$

$ZETA(14,12) = 1/1556755200\ B^{2}$

$ZETA(14,14) = 1/87178291200$

$TAU(12) = 400816/25727625\ B^{2} + 34023424/28704375\ B^{4} + 4417024/127575\ B^{6} + 971264/2025\ B^{8} + 28672/9\ B^{10} + 8192\ B^{12} + 57122836/1917454163625$

$ZETA(15,0) = 16/9577693125\ B + 11874263764432/45299854615640625\ B^{3} + 16586161216/218813450625\ B^{5} + 6957056/1366875\ B^{7} + 126795776/893025\ B^{9} + 3909632/2025\ B^{11} + 114688/9\ B^{13} + 32768\ B^{15}$

$ZETA(15,2) = 316/8300667375\ B + 21270108064/33186706678125\ B^{3} + 591122992/6630710625\ B^{5} + 15903232/4465125\ B^{7} + 860672/14175\ B^{9} + 63488/135\ B^{11} + 4096/3\ B^{13}$

$ZETA(15,4) = 257/2341213875\ B + 291182629/1005657778125\ B^{3} + 3207104/147349125\ B^{5} + 110416/212625\ B^{7} + 4736/945\ B^{9} + 256/15\ B^{11}$

$ZETA(15,6) = 2083/22986463500\ B + 5646296/120678933375\ B^{3} + 39712/21049875\ B^{5} + 208/8505\ B^{7} + 32/315\ B^{9}$

$ZETA(15,8) = 271/10216206000\ B + 14501/4378374000\ B^{3} + 37/561330\ B^{5} + 1/2835\ B^{7}$

$ZETA(15,10) = 83/29189160000\ B + 31/291891600\ B^{3} + 1/1247400\ B^{5}$

$ZETA(15,12) = 1/10007712000\ B + 1/778377600\ B^{3}$

$ZETA(15,14) = 1/653837184000\ B$

TABLE 2 (Cont'd)

$$ZETA(16,0) = 8308816/9587270818125\ B^{2} + 2028886032512/1006663435903125\ B^{4} + 37696/121275\ B^{6} + 1058987008/66976875\ B^{8} + 66615296/178605\ B^{10} + 1015808/225\ B^{12} + 81920/3\ B^{14} + 65536\ B^{16}$$

$$ZETA(16,2) = 343946452/67110895726875\ B^{2} + 23783616952/6637341335625\ B^{4} + 410450752/1326142125\ B^{6} + 131771264/13395375\ B^{8} + 6178304/42525\ B^{10} + 137216/135\ B^{12} + 8192/3\ B^{14}$$

$$ZETA(16,4) = 371341/81942485625\ B^{2} + 9501626/7449316875\ B^{4} + 1932704/29469825\ B^{6} + 91328/70875\ B^{8} + 384/35\ B^{10} + 512/15\ B^{12}$$

$$ZETA(16,6) = 350201/268175407500\ B^{2} + 6712879/40226311125\ B^{4} + 6964/1403325\ B^{6} + 464/8505\ B^{8} + 64/315\ B^{10}$$

$$ZETA(16,8) = 34277/214540326000\ B^{2} + 15923/1641890250\ B^{4} + 17/112266\ B^{6} + 2/2835\ B^{8}$$

$$ZETA(16,10) = 277/29189160000\ B^{2} + 1/3891888\ B^{4} + 1/623700\ B^{6}$$

$$ZETA(16,12) = 19/70053984000\ B^{2} + 1/389188800\ B^{4}$$

$$ZETA(16,14) = 1/326918592000\ B^{2}$$

$$ZETA(16,16) = 1/20922789888000$$

$$TAU(14) = 10855578713152/5033317179515625\ B^{2} + 18818625536/72937816875\ B^{4} + 107117056/9568125\ B^{6} + 209985536/893025\ B^{8} + 1753088/675\ B^{10} + 131072/9\ B^{12} + 32768\ B^{14} + 49292224/28761812454375$$

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