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Résumé

Les caractéristiques de déformation élastique de divers types de glace naturelle ont été déterminées à partir des cinq constantes d'élasticité de la glace monocristalline. L'analyse permet de comparer les modules d'élasticité mesurés de divers types de glace. Elle propose aussi des équations simples à deux termes visant à calculer, pour divers types de glace courants, la dépendance des modules d'Young et de rigidité sur la température et la direction des charges, et pour la glace granuleuse, elle propose une relation simple entre le coefficient de Poisson et la température.

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ELASTICITY OF NATURAL TYPES OF POLYCRYSTALLINE ICE

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ABSTRACT

Elastic deformation characteristics of a number of natural ice types have been determined using five single crystal compliances. The analysis provides a basis on which measured values of the elastic moduli of various types of ice can be compared. It also provides simple two-term equations for the temperature dependence of Young's modulus and rigidity modulus for common ice types and loading directions and, for granular ice, a simple relation for the temperature dependence of Poisson's ratio.

INTRODUCTION

To date many problems in ice engineering have been dealt with elastically in an empirical manner. The questions often raised are: What is Young's modulus of ice? What is Poisson's ratio of ice? These are crucial questions. Answers are required before any elastic or viscoelastic solutions can be applied. Yet little attention has been paid to the subject itself – the elasticity of polycrystalline ice, particularly the response of the various types of polycrystalline ice encountered in nature for various types of loading conditions. In comparison, however, excellent studies have been made on the elastic response of a single crystal of ice. The reasons for the apparent lack of information on polycrystalline ice are the difficulties in separating pure elastic response from other concomitant viscoelastic responses of the material at working temperatures and loading conditions important to real life engineering problems with ice.

Elasticity, nonetheless, plays a major role in shaping the viscoelastic response of ice. An effort is made herein to explain and calculate the elastic response, including anisotropic response, of various types of ice. Knowledge of the elastic properties of a single crystal of ice and the microstructural differences of the different types of polycrystalline ice is the foundation of this work.

Since the analyses to be carried out are based on single crystal elastic response, a brief summary is given in the first part of this paper on how a single crystal, in general, and a hexagonal crystal, in particular, respond to an applied load.

The second part introduces the information available on single crystals of ice. Methods are then described for calculating the polycrystalline response from the major crystallographic features of ice commonly observed in nature and data available on single crystals. Since the equations and methods of computations are rather involved, the calculated results are presented in the form of simple empirical equations. These formulas describe the temperature dependence of Young's modulus and the rigidity modulus.

ICE SINGLE CRYSTAL ELASTICITY

Hooke's elastic theory assumes a linear relationship between stress, σ_{ij}, and strains, ϵ_{kl}, and according to tensorial notation it is presented as:

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} \qquad (1)$$

or its inverse form:

$$\epsilon_{ij} = S_{ijkl}\sigma_{kl} \qquad (2)$$

where C_{ijkl} are called the elastic or stiffness constants and S_{ijkl} the compliances.

In a rectangular coordinate system $(i,j = 1,2,3)$

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symmetry considerations in stress, $\sigma_{ij}=\sigma_{ji}$, and strain, $\epsilon_{ij}=\epsilon_{ji}$, result in only six independent components of stress and strain, and the 81 components of the stiffness constant, C_{ijkl}, or the compliance, S_{ijkl}, reduce to 21. A simplified relation in the form of a square matrix of the order six can be used to represent the relationship between the six components of stress and strains (Nye, 1957; Sneddon and Berry, 1958):

$$\sigma_i = C_{ij}\epsilon_j \qquad i,j=1 \text{ to } 6 \tag{3}$$

$$\epsilon_i = S_{ij}\sigma_j \qquad i,j=1 \text{ to } 6 \tag{4}$$

The reversibility of the elastic strains dictate the symmetry relation, $C_{ij}=C_{ji}$ and $S_{ij}=S_{ji}$. Because of higher crystallographic symmetry, the number of independent coefficients decreases to nine for an orthorhombic crystal, five for a hexagonal crystal and three for a cubic crystal. The number of constants is reduced to only two for a non-crystalline or an amorphous material like ordinary window glass.

Ice belongs to the family of hexagonal crystals which includes cadmium, magnesium, titanium and zinc. For hexagonal crystal, there are twelve non-zero coefficients in the matrix C_{ij} and S_{ij} (Nye, 1957; Sneddon and Berry, 1958). Of these, only five are completely independent. These are C_{11}, C_{12}, C_{13}, C_{33} and C_{44} for the stiffness constants and S_{11}, S_{12}, S_{13}, S_{33} and S_{44} for the compliances as illustrated in Fig. 1.

What happens to a single crystal of ice when a load is applied externally at an arbitrary angle? This can be determined from the dependence on direction modulus of the elasticity, E, and the shear modulus, G. For a hexagonal crystal this dependence is given by (Bass et al., 1957; Nowick and Berry, 1972; Kelly, 1981):

$$E_\phi = 1/\{S_{11}(1-l^2)^2 + S_{33}l^4 + (2S_{13}+S_{44})l^2(1-l^2)\} \tag{5}$$

and:

$$G_\phi = 1/[S_{44} + \{S_{11} - S_{12} - (S_{44}/2)\}(1-l^2) + 2(S_{11}+S_{33}-2S_{13}-S_{44})l^2(1-l^2)] \tag{6}$$

where $l=\cos\phi$ is the direction cosine, ϕ is the angle between the direction of the stress and the axis of crystallographic symmetry, known as the c-axis.

$$S_{ij} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{13} & 0 & 0 & 0 \\ S_{13} & S_{13} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(S_{11}-S_{12}) \end{bmatrix}$$

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(C_{11}-C_{12}) \end{bmatrix}$$

Fig. 1. Matrices for S_{ij} and C_{ij} for a hexagonal ice crystal. Non-zero equal quantities are joined by lines to identify clearly the five independent components.

Equations 5 and 6 show that E and G depend only on the angle from the c-axis which is normal to the basal plane – the plane that contains the crystallographic a-axis. The moduli are, therefore, isotropic in the basal plane, $l=0$, irrespective of the orientation of the a-axis.

MEASUREMENT OF THE ELASTIC CONSTANTS OF ICE

Several measurements have been made of the elastic constants of ice. A list of those made prior to 1964 was given in Fletcher (1970). To date, the most complete and probably the most accurate values of the elastic constants of single crystal ice are those of Dantl (1969). He determined their frequency dependence in the range of 5–190 MHz and their temperature dependence from the melting

TABLE 1

The compliances at melting point, T_m (K) and other constants for ice (from Dantl, 1969)

$S_{ij}(T)$ ($m^2 MN^{-1}$)	S^0_{ij} ($m^2 MN^{-1}$)	A	B	Error in $S_{ij}(T)$ (%)
S_{11}(T)	10.40×10^{-11}	1.070×10^{-3}	1.87×10^{-6}	±1
S_{33}(T)	8.48×10^{-11}	1.405×10^{-3}	4.66×10^{-6}	±1
S_{44}(T)	33.42×10^{-11}	1.505×10^{-3}	4.04×10^{-6}	±1
S_{12}(T)	-4.42×10^{-11}	0.463×10^{-3}	2.06×10^{-6}	±6
S_{13}(T)	-1.89×10^{-11}	1.209×10^{-3}	6.15×10^{-6}	±20

point, T_m, of 273 K to 133 K (0 to -140°C) by a supersonic pulse-echo method and a double-pulse interference technique. Ice was grown in the laboratory at a growth velocity of 0.2 μm s^{-1} to obtain as pure crystals as possible. The final machined specimens were rather large and were in the form of cylinders, 100 mm in diameter and 250 mm in length. The long dimensions of the specimens were parallel to the *c*-axis within ±0.6°C. No frequency dependence of the elastic moduli was observed in the range studied. The temperature dependence was determined at 30 MHz in intervals of 5K and the results can be presented in the form:

$$S_{ij}(T)=S^0_{ij}\{1+A(T-T_m)+B(T-T_m)^2\} \qquad (7)$$

or:

$$C_{ij}(T)=C^0_{ij}\{1+A'(T-T_m)+B'(T-T_m)^2\} \qquad (8)$$

where T is in Kelvin and S^0_{ij} and C^0_{ij} are the values at the melting point, T_m, and A and B are constants. The compliances, determined by Dantl and used in this paper, are given in Table 1.

GRANULAR ICE

Granular ice – snow ice or consolidated frazil slush ice – is a common form of natural ice that is produced by the freezing of snow or other ice particles saturated with water. This type of ice can be seen in the ice covers of rivers, lakes or oceans, and usually in the upper sections of glaciers. The grains can be rounded, equiaxed or angular and the *c*-axis of the grains are usually randomly oriented in all planes (Fig. 2). The density of this ice can be high, within a few percent of that of single crystals. Macroscopically, this type of ice is isotropic and its elastic properties can be expressed by two independent constants, S^e_{11} and S^e_{12}, the superscript, e, is used to indicate that they are derived quantities.

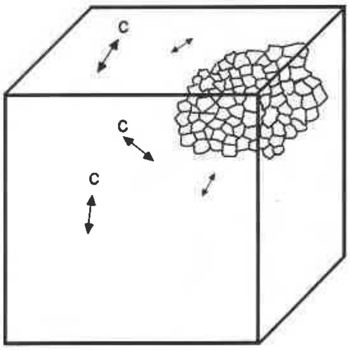

Fig. 2. Granular ice.

Many attempts have been made to derive S^e_{11} and S^e_{12} for a polycrystalline mass having randomly oriented grains from the single crystal elastic constants (Voigt, 1910; Reuss, 1929). Some averaging principles were followed in these derivations. The most appropriate for application to ice are the results obtained for a hexagonal system (Markham, 1962) given by:

$$S^e_{11}=(8S_{11}+3S_{33}+4S_{13}+2S_{44})/15 \qquad (9)$$

$$S^e_{12}=(S_{11}+S_{33}+5S_{12}+8S_{13}-S_{44})/15 \qquad (10)$$

The Young's modulus, E, Poisson's ratio, μ, and the shear or rigidity modulus, G, are given by:

$$E=1/S^e_{11}=15/(8S_{11}+3S_{33}+4S_{13}+2S_{44}) \qquad (11)$$

$$\mu = -S^e_{12}/S^e_{11}$$

$$= -(S_{11}+S_{33}+5S_{12}+8S_{13}-S_{44})/ \quad (12)$$

$$(8S_{11}+3S_{33}+4S_{13}+2S_{44})$$

$$G = 1/S^e_{44} = E/[2(1+\mu)] \quad (13)$$

$$= 1/[2(S^e_{11}-S^e_{12})]$$

Calculations using Eqs. 11, 12 and 13, and the values of the compliances given by the Eq. 7 and Table 1 are plotted in Fig. 3 for the temperature range relevant to most engineering requirements. It may be noticed that all the quantities increase slightly, with the decrease in temperature, in an almost linear fashion. Young's modulus increases by 5% in the range 0°C to −38°C from 8.93 to 9.39 GN m^{-2}. The shear modulus also increases by about 5% in this range. Poisson's ratio increases by about 1% in the same temperature range. For engineering purposes, these temperature dependencies may be described by the following relations:

$$E_T = E_{T_m} + c(T_m - T) \quad (14a)$$

where: $E_{T_m} = 8.93$ GN m^{-2} and $c = 1.2\times10^{-2}$ GN $m^{-2}K^{-1}$

$$\mu_T = \mu_{T_m} + d(T_m - T) \quad (14b)$$

where: $\mu_{T_m} = 0.308$ and $d = 7\times10^{-5}K^{-1}$

$$G_T = G_{T_m} + e(T_m - T) \quad (14c)$$

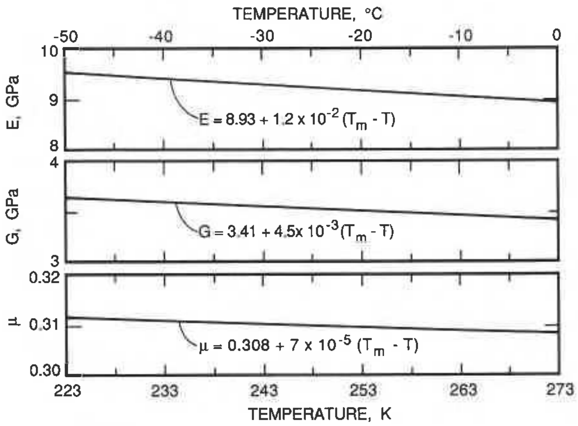

Fig. 3. Temperature dependence of Young's modulus, E, rigidity modulus, G, and Poisson's ratio, μ, of granular ice; T_m is the melting point.

where: $G_{T_m} = 3.41$ GN m^{-2} and $e = 4.5\times10^{-3}$ GN $m^{-2}K^{-1}$

Note that the first terms in Eq. 14 provide the elastic response of the bottom layer of floating ice covers because the ice-water interface is always at or near T_m. However the bottom layers of ice covers and, in fact, the bulk of ice covers, are very rarely granular in nature; anisotropic ice such as columnar-grained ice is commonly present.

COLUMNAR-GRAINED ICE – *C*-AXIS VERTICAL

The bulk of an ice cover initially formed under calm conditions exhibits anisotropy even on a macroscopic scale. The *c*-axis of the grains tend to be in the vertical plane (Fig. 4) and hence parallel to the growth direction which is normal to the plane of the ice cover. The grains could be orders of magnitude larger than those in granular snow ice discussed above. This type of ice is classified as S1 ice by Michel and Ramseier (1971).

Since the *c*-axis of all the grains are more or less in the vertical plane, this ice exhibits mechanical anisotropy similar to a single crystal. Equations 5 and 6 can be used respectively for E and G provided ϕ represents the angle from the vertical axis. The response will vary with the angle from the vertical direction but will be isotropic in the horizontal plane.

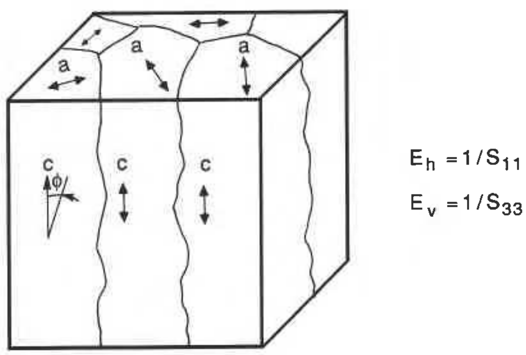

Fig. 4. Columnar-grained ice with *c*-axis in the vertical plane or parallel to the direction of growth (type S1).

If μ' is the effective Poisson's ratio, then an approximate quantity can be obtained if it is assumed (which is strictly correct for truly isotropic material) that $G=E/[2(1+\mu)]$, so that from Eqs. 5 and 6:

$$\mu' = (E/2G)-1$$
$$= [S_{44}+\{S_{11}-S_{12}-(S_{44}/2)\}$$
$$(1-l^2)+2(S_{11}+S_{33}-2S_{13}-S_{44})$$
$$l^2(1-l^2)]/2[S_{11}(1-l^2)^2+S_{33}l^4$$
$$+(2S_{13}+S_{44})l^2(1-l^2)]-1 \quad (15)$$

For many situations of practical interest, stress is imposed in the horizontal plane. This occurs when an ice cover is used for transportation or recreational purposes or if it reacts against a structure such as docks or retaining walls. If $E_{(h)}$ and $G_{(h)}$ and $\mu'_{(h)}$ are the material constants in the horizontal plane or the plane of the ice cover, then Eqs. 5, 6 and 15 give for $l=0$ ($\phi=90°$):

$$E_{(h)}=1/S_{11} \quad (16a)$$

$$G_{(h)}=1/[S_{44}/2+S_{11}-S_{12}] \quad (16b)$$

$$\mu'_{(h)}=(S_{44}-2S_{11}-2S_{12})/S_{44} \quad (16c)$$

where the symbol ( ) is used to indicate a plane.

Mechanically it is a transversely isotropic material with isotropy in the horizontal plane. It should be mentioned here that Young's modulus in the vertical direction will be given by $E_{\langle v\rangle}=1/S_{33}$ (Eq. 5 for $l=1$) and the shear modulus in any vertical plane is given by $G_{(v)}=1/S_{44}$ (Eq. 6 for $l=1$); the symbol ⟨ ⟩ is used to indicate a direction, as opposed to ( ), used for a plane. Since S_{33} is significantly lower than S_{11}, $E_{\langle v\rangle}$ is significantly higher than $E_{(h)}$. At 263 K the values are 11.96 GN m^{-2} and 9.72 GN m^{-2} respectively – a 23% difference. $G_{(v)}$, however, is lower than $G_{(h)}$ and at 263 K it is 2.99 and 3.17 GN m^{-2} respectively – a difference of 6%.

Calculations using Eqs. 16, 7 and Table 1 are plotted in Fig. 5 for Young's modulus and the rigidity modulus. It can be seen that the moduli exhibit the lowest (or highest) value at an angle of about 40° from the horizontal plane. Not shown in the figure are the calculations on $\mu'_{(h)}$. They are rather high (=0.51), indicating an overestimation and, hence, the limited applicability of Eqs. 15 and 16c. Experimental data are required to examine the validity of these equations.

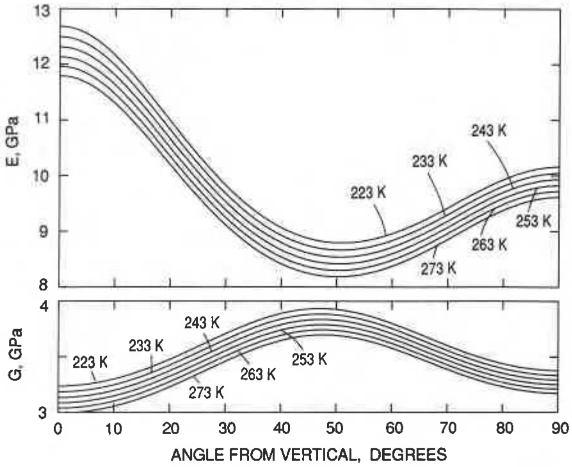

Fig. 5. Dependence of the elastic moduli of S1 type ice on temperature and orientation from normal to the plane of the ice cover.

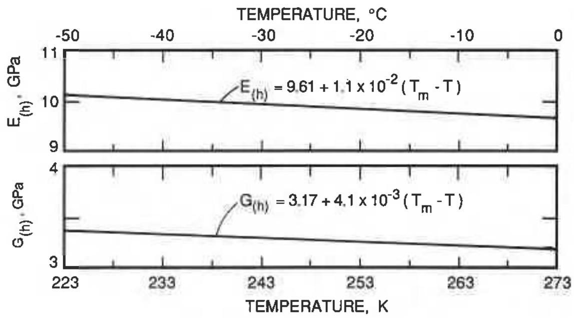

Fig. 6. Temperature dependence of the elastic moduli of S1 type ice in the plane of ice cover; T_m is the melting point.

Figure 6 shows an increase in the moduli with the decrease in temperature. Simple relations can be suggested for engineering applications:

$$E_{(h)'T}=E_{(h)'T_m}+f(T_m-T) \quad (17a)$$

where: $E_{(h)'T_m}=9.61$ GN m^{-2}, and $f=1.1\times10^{-2}$ GN m^{-2} K^{-1} and:

$$G_{(h)'T}=G_{(h)'T_m}+g(T_m-T) \quad (17b)$$

where: $G_{(h)'T_m}=3.17$ GN m^{-2} and $g=4.1\times10^{-3}$ GN m^{-2} K^{-1}.

COLUMNAR-GRAINED ICE – *C*-AXIS HORIZONTAL AND RANDOM

The bulk of an ice cover in large bodies of water often has a columnar-grained structure with the length of the grains parallel to the growth direction (the vertical direction) and the *c*-axis of the grains tending to be randomly oriented in the horizontal plane (Fig. 7). In this type of ice, classified as S2 type (Michel and Ramseier, 1971), planes parallel to the length of the columns also contain the basal planes of all the grains. For a vertical load, therefore, $l=0$ for each grain. Equation 5 gives, for this condition, the modulus in the vertical direction:

$$E_{\langle v \rangle}=1/S_{11} \quad (18)$$

Equations 16a and 18 show that $E_{(h)}$ for S1 ice and $E_{\langle v \rangle}$ for S2 ice are the same. Hence, the temperature dependence of $E_{\langle v \rangle}$ is also given by Eq. 17a:

$$E_{\langle v \rangle}(\text{S2 ice})=E_{(h)}(\text{S1 ice})$$
$$=E_{(h)'T_m}+f(T_m-T)$$
$$=9.61+1.1\times10^{-2}(T_m-T) \quad (19)$$

Random orientation of the *c*-axis of the grains in the horizontal plane means $1 \geq l \geq 0$ for any direction in this plane. Young's modulus for this plane, $E_{(h)}$ is given by the average value:

$$E_{(h)}=\int_0^{\pi/2} E_\phi \mathrm{d}\phi \Big/ \int_0^{\pi/2} \mathrm{d}\phi$$
$$=(2/\pi)\int_0^{\pi/2} [1/\{S_{11}(1-l^2)^2+S_{33}l^4 + (2S_{13}+S_{44})l^2(1-l^2)\}]\mathrm{d}\phi \quad (20)$$

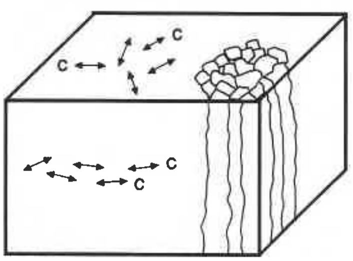

Fig. 7. S2 ice structure.

by substitution of E_ϕ from Eq. 5.

Similarly, the rigidity modulus, $G_{(h)}$, in the horizontal plane is given by:

$$G_{(h)}=\int_0^{\pi/2} G_\phi \mathrm{d}\phi \Big/ \int_0^{\pi/2} \mathrm{d}\phi$$
$$=(2/\pi)\int_0^{\pi/2} \{1/[S_{44}+\{S_{11}-S_{12}-(S_{44}/2)\}(1-l^2) + 2(S_{11}+S_{33}-2S_{13}-S_{44})l^2(1-l^2)]\}\mathrm{d}\phi \quad (21)$$

by substitution of G_ϕ from Eq. 6.

If $\mu'_{(h)}$ is the effective Poisson's ratio in the horizontal plane, then one can estimate its approximate value from the usual relation (for isotropic material):

$$\mu'_{(h)}=(E_{(h)}/2G_{(h)})-1 \quad (22)$$

Numerical integrations were used to calculate $E_{(h)}$, $G_{(h)}$, and $\mu'_{(h)}$ using values of the coefficients given by Eq. 7 and Table 1. The results are plotted in Fig. 8 for $E_{(h)}$ and $G_{(h)}$. A nearly linear temperature dependence of the elastic moduli can be seen and, for engineering applications, may be stated as:

$$E_{(h)'T}=E_{(h)'T_m}+h(T_m-T) \quad (23a)$$

where: $E_{(h)'T_m}=9.39$ GN m^{-2} and $h=1.3\times10^{-2}$ GN m^{-2}K^{-1} and:

$$G_{(h)'T}=G_{(h)'T_m}+i(T_m-T) \quad (23b)$$

where: $G_{(h)'T_m}=3.37$ GN m^{-2} and $i=4.7\times10^{-3}$ GN m^{-2}K^{-1} and:

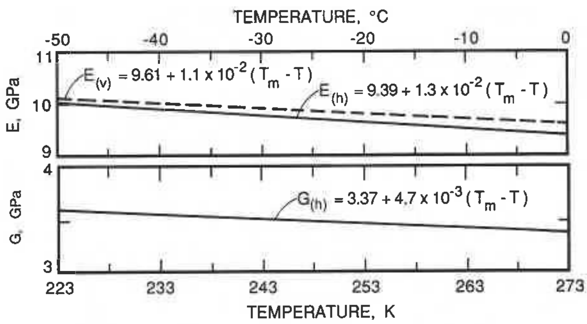

Fig. 8. Temperature dependence of the elastic moduli of S2 type ice in the plane of ice cover and normal to it.

$$\mu'_{(h)} = \mu'_{(h)'T_m} + j(T_m - T) \quad (23c)$$

where: $\mu'_{(h)'T_m} = 0.39$ and $j = 7\times10^{-5}\mathrm{K}^{-1}$

The value of $\mu'_{(h)}$ was 0.393 at the melting point and increased by 1% to 0.396 at 223 K. These values of $\mu'_{(h)}$, though more "believable" than those given by Eq. 16c for S1 ice, are also overestimated quantities, as will be discussed later, and should be used with caution. This is because of the nonapplicability (strictly speaking) of Eq. 22 for anisotropic materials.

COLUMNAR-GRAINED ICE – *C*-AXIS HORIZONTAL AND ORIENTED

Landfast sea ice covers generally consist of columnar-grained material with the *c*-axis of the grains in the horizontal plane. Often a strong orientation of the *c*-axis of the grains is found in the plane of the ice cover (Fig. 9). This preferred orientation was linked by Weeks and Gow (1978) to the current in the water under the ice cover. This observation was confirmed by Nakawo and Sinha (1984) who also provided microstructural details as a function of depth and the growth history. The oriented sea ice structure has also been reported in a multi-year sea ice floe in the high Arctic (Sinha, 1987) and could be classified as S3 type following the system of Michel and Ramseier (1971). It is important therefore to examine the elastic response of this ice.

As for the vertical direction, S3 ice is similar to S2 ice. All the grains have their basal planes oriented vertically. Equation 19, therefore, gives Young's modulus and its temperature dependence for S3 ice in the vertical direction. In the horizontal plane, however, these two types of ice are distinguishable mechanically because of the anisotropy in the fabric of S3 ice. Since the crystals are oriented within a scatter angle, θ from the mean *c*-axis or the direction of the current, the macroscopic behaviour in any direction making an angle ϕ from the mean *c*-axis (or the current) will be given by the average value within the angle $\phi\pm\theta$, so that Eqs. 20 and 21 respectively take the form:

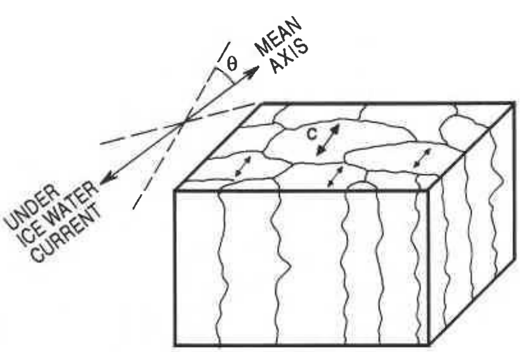

Fig. 9. S3 sea ice structure.

$$E_{(h)} = \int_{\phi-\theta}^{\phi+\theta} E_\phi \mathrm{d}\phi \Big/ \int_{\phi-\theta}^{\phi+\theta} \mathrm{d}\phi$$

$$= \frac{1}{2\theta}\int_{\phi-\theta}^{\phi+\theta} [1/\{S_{11}(1-l^2)^2 + S_{33}l^4 + (2S_{13}+S_{44})l^2(1-l^2)\}]\mathrm{d}\phi \quad (24)$$

and:

$$G_{(h)} = \int_{\phi-\theta}^{\phi+\theta} G_\phi \mathrm{d}\phi \Big/ \int_{\phi-\theta}^{\phi+\theta} \mathrm{d}\phi$$

$$= \frac{1}{2\theta}\int_{\phi-\theta}^{\phi+\theta} \{1/[S_{44} + \{S_{11} - S_{12} - (S_{44}/2)\} (1-l^2) + 2(S_{11}+S_{33}-2S_{13}-S_{44})^2 (1-l^2)]\}\mathrm{d}\phi \quad (25)$$

A running mean using Eqs. 24 and 25, respectively, describes the angular dependence of $E_{(h)}$ and $G_{(h)}$. Calculations are presented in Fig. 10 for a few scatter angles. It shows that the response of S3 ice approaches, as it should, that of S2 ice as the scatter angle increases.

DISCUSSION

As mentioned in the introduction, there are very limited experimental data on the nature of true elastic response of pure (bubble-free) polycrystalline ice. Most measurements, due to the presence of inclusions and the static nature of the techniques used, suffered from the fact that deformations other than elastic strains were also measured. However, experimental values of 9.9 GN m^{-2} reported by Gold (1958) and 9.5±0.3 GN m^{-2} measured by Sinha (1978) at 233 K for horizontal loading of inclusion free S2 ice provide some support for the theory presented above.

Nakaya's (1959) data, on the dynamic Young's modulus of ice from a tunnel drilled into the Greenland ice cap, show the modulus to decrease from about 9.2 to 7.5 GN m^{-2} for a decrease in density from 917 to 905 kg m^{-3}. The results are rather scattered. There is, as yet, no reliable measurement of the elastic response of pure granular ice. It is, therefore, not possible to establish whether the E modulus of this ice is slightly, but measurably, lower than that of columnar-grained ice.

The seismic resonance experiments of Langleben and Pounder (1963) using sea ice indicated that the average response does not exhibit any significant dependence either on the orientation or the type of ice. They found that $E=(10-0.035\ v)$ GN m^{-2}, where v is the brine volume in parts per thousand (‰), fit their results on natural and built-up Arctic sea ice in winter conditions. For sea ice at −10°C with a salinity of 4‰ or brine volume of 20‰ (Frankenstein and Garner, 1967), the equation gives Young's modulus of 9.3 GN m^{-2} which agrees well with the theory (Fig. 10) and shows that the macroscopic elastic response of Arctic sea ice can be predicted reasonably well.

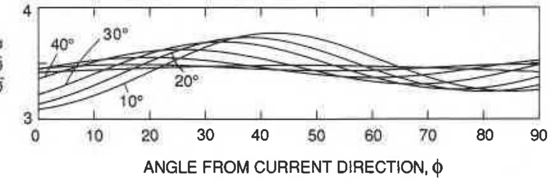

Fig. 10. Angular dependence of the moduli of S3 type sea ice in the plane of the ice cover for selected amounts of preferred orientation at 263 K.

Langleben and Pounder (1963) also reported that their value for the Poisson's ratio was relatively constant for sea ice, apparently independent of temperature in the range of −3.6 to −15°C. They suggested a value of 0.295 which bears good agreement with the value of 0.29 reported by Peschansky (1957). For S2 ice loaded perpendicular to the columns, Gold's (1958) experiments indicated values in the range of 0.31–0.54 for the ratio of transverse to longitudinal strains as the strain increased with time during creep tests. Since these values are for increasing strain levels, the lowest value, corresponding to the lowest strain levels during the initial period of loading, could be considered as the Poisson's ratio because contributions of non-elastic strains to the total strain would be minimal during this period. Poisson's ratio in the range of 0.31–0.32 was obtained for horizontal loading of S2 ice at −20°C and a loading frequency of about 100 Hz (Sinha, 1988). In view of all the above observations it is fair to say that the agreement is excellent between these measurements and the calculated results of 0.31 for Poisson's ratio and its negligible temperature dependence given in Eq. 14b in the case of granular ice. Direct measurements on S2 ice at relatively high frequencies also support the idea that Eq. 23c is not valid in a strict sense, although it gives values which are 'believable'.

CONCLUSIONS

Methods have been developed for calculating the dependence of Young's modulus (E) on temperature and orientation, the shear or rigidity modulus, G, and Poisson's ratio of polycrystalline ice – granular and columnar-grained – on the basis of single crystal data. It is shown that the elastic properties of various types of ice, in spite of large differences in fabric, are very similar at a constant temperature. Elastic response of transversely isotropic, columnar-grained, S2 ice is nearly isotropic; E in the vertical direction is only about two percent greater than that in the horizontal plane. Columnar grained, S1 type ice shows the maximum anisotropy; E in the vertical direction is 23% larger than that in the horizontal plane. The response of this ice in the horizontal plane, however, is the same as that of the S2 ice in the vertical direction. S3 type oriented sea ice shows behaviour similar to S2 ice with slight anisotropy in the horizontal plane that depends on the degree of orientation. Granular ice shows the least stiffness. Its Young's modulus is 5% lower than that of S2 ice in the horizontal plane and 7% lower than that of S1 ice in the same plane. The Poisson's ratio of granular ice is 0.31 and it increases by about 1% in the temperature range 233–273 K, the range important to engineering needs. In the same temperature range, Young's modulus of all types of ice increases by about 5% as the temperature decreases.

The most useful parts of this exercise are the development of simple equations, readily applicable to engineering needs, that give the temperature dependence of E and G (and μ for granular ice).

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