Study of ground pressures

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PREFACE

The National Research Council, through its Division of Building Research, is pleased to be able to publish this further translation prepared by the staff of the University of Alberta. The translation has been kindly made available through the cooperation of Professor I. F. Morrison of the Department of Civil and Municipal Engineering and Professor T. Patching of the Department of Mining Engineering.

The subject of ground pressures has long been one of singular importance in civil and mining engineering. The paper now translated is a most significant contribution to the literature. It is therefore a privilege to be able to publish this excellent translation.

The translation was prepared by Mr. Boris Korun and Mr. William Kochan who graduated from the University of Alberta in mining engineering in 1953. Mr. Kochan was born in Alberta, coming directly to the University from his high school. He was responsible for the final version of the translation. The original translation from the German was done by Mr. Korun who was born in Yugoslavia. He spent some time at the University of Bologna in Italy before coming to Canada where he pursued his studies at the University of Alberta. Mr. Kochan is now working for Imperial Oil Limited and Mr. Korun has been engaged in exploration work in northern Canada.

The Division of Building Research is again indebted to Professor I. F. Morrison for bringing this translation to its attention and for facilitating its publication in this form. Professor Morrison spent much of his own time in work on the translation in keeping with his long standing interest in the development of engineering research work in Canada.

R. F. Legget,
Director.

SUPPLEMENTARY NOTE

Just prior to the reproduction of this translation it was found that this paper by Mr. Fenner was of far greater interest than the Division of Building Research has suspected. It is known to those who, in Canada, are concerned with the problems of rock bursts in mines. Through the Division of Fuels of the Mines Branch of the Department of Mines and Technical Surveys, further study of the problems discussed by Fenner has been made at McGill University. It is therefore hoped that this translation from the original German version will prove to be of service in several fields of research in the Canadian economy.

R.F.L.

OTTAWA

9 February 1955

NATIONAL RESEARCH COUNCIL OF CANADA

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Title: Study of ground pressures.

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Author: R. Fenner.

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Translators: W. Kochan and B. Korun.

STUDY OF GROUND PRESSURES

In the investigation of ground pressures so far, the problem has been handled from two different points of view. Some investigators base their work on personal experiences and observations, whereas others use laws such as Hooke's law and mathematical formulae to obtain theoretical results.

Literature on mining contains many statements and theoretical equations(1) which are difficult to subject to a common viewpoint. In the following work, the writer limits himself to pressures existing around small openings, such as access tunnels and shafts* and tries to develop theories based on the elasticity and plasticity of the ground without contradicting the results obtained by valuable observations.

It is possible to solve the problem of ground pressures by studying an ideal state and setting up relationships which apply to an absolutely homogeneous medium in which the stress conditions occur. This study must be limited to the range in which Hooke's law is valid, that is, within the flow limits. Although the plastic relationships are also valid beyond the flow limits, the mathematics involved in solving the problem of an extreme case is a very involved and tedious procedure. This condition is known not to occur in a mine since the flow conditions are much too slow. For this reason, it is unnecessary to discuss the stress conditions involved in large openings, although the theory is the same as that for small openings.

Stress Conditions in Undisturbed Ground

Stress Conditions in an Elastic Medium

By definition, the larger of the two principal stresses in undisturbed ground is in the vertical direction and may be expressed by the equation,

$$p_z = \gamma h \tag{1}$$

where p_z is the vertical ground pressure, γ is the specific weight of the ground, and h the depth of the area under investigation.

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* The literature on the many observations and measurements on the surface is not given here because they relate to large cavities.

Because the horizontal pressure in the o x and o y directions counteract deformations, the strains in the x, y, and z directions can be expressed by the following formulae.

$$\left.\begin{aligned} &\frac{\partial\xi}{\partial x} = \frac{\partial\xi}{\partial y} = \frac{\partial\xi}{\partial z} = \frac{\partial\eta}{\partial x} = \frac{\partial\eta}{\partial y} = \frac{\partial\eta}{\partial z} = \frac{\partial\zeta}{\partial x} = \frac{\partial\zeta}{\partial y} = 0 \\ &\qquad\qquad\qquad\frac{\partial\zeta}{\partial z} \neq 0 \end{aligned}\right\} \quad (2)$$

According to the theories of elasticity[2] the principal stresses may be expressed by the following equations:

$$\left.\begin{aligned} \sigma_x &= 2\,G\left(\frac{\partial\xi}{\partial x} + \frac{e}{m-2}\right) \\ \sigma_y &= 2\,G\left(\frac{\partial\eta}{\partial y} + \frac{e}{m-2}\right) \\ \sigma_z &= 2\,G\left(\frac{\partial\zeta}{\partial z} + \frac{e}{m-2}\right) \end{aligned}\right\} \quad (3)$$

where G is the modulus of rigidity, e is the volume expansion, and m is Poisson's number. By combining equations (2) and (3) and letting $\sigma_z = p$, the following relationship of the stress conditions may be obtained.

$$\sigma_x = \sigma_y = \frac{p}{m-1} \quad (4)$$

By taking the formulae valid for the shearing stresses

$$\left.\begin{aligned} \tau_{xy} &= \tau_{yx} = G\left(\frac{\partial\xi}{\partial y} + \frac{\partial\eta}{\partial x}\right) \\ \tau_{xz} &= \tau_{zx} = G\left(\frac{\partial\xi}{\partial z} + \frac{\partial\zeta}{\partial x}\right) \\ \tau_{yz} &= \tau_{zy} = G\left(\frac{\partial\zeta}{\partial y} + \frac{\partial\eta}{\partial z}\right) \end{aligned}\right\} \quad (5)$$

the following result may be obtained:

$$\tau_{xy} = \tau_{yx} = \tau_{xz} = \tau_{zx} = \tau_{yz} = \tau_{zy} = 0.$$

This proves that the three normal stresses are the principal stresses.

With reference to Fig. 2, which represents the stress conditions in a given plane, a geometrical analysis of the situation will yield these equations.

$$\left.\begin{aligned} \sigma_x \sin\phi &= n \sin\phi - \tau\cos\phi \\ \sigma_z \cos\phi &= n \cos\phi + \tau\sin\phi \end{aligned}\right\} \qquad (6)$$

Equations (6) show that both components

$$x = \sigma_x \sin\phi \quad \text{and} \quad z = \sigma_z \cos\phi$$

satisfy the resultant R in the equation:

$$\frac{x^2}{\sigma_x^2} + \frac{z^2}{\sigma_z^2} = 1 \qquad (7)$$

In the above stress conditions, the o y axis has the same value as the o x axis, i.e., $\sigma_x = \sigma_y$ in the equations involved.

$$\frac{x^2 + y^2}{\sigma_x^2} + \frac{z^2}{\sigma_z^2} = 1 \qquad (8)$$

Therefore, the stress ellipsoid is rotational with axis o z.

$$\sigma_z = p; \ \sigma_x = \frac{p}{m - 1}; \ \sigma_y = \frac{p}{m - 1}.$$

These equations are valid as long as the shearing stresses do not go beyond the limit of proportionality of the rock concerned.

For practical reasons, the stress condition may be expressed in polar form more conveniently. In Fig. 3, σ_t and σ_r

are the normal stresses and τ is the shearing stress. The angle ϕ in Fig. 2 corresponds to angle ϕ in Fig. 3. By the use of equations (6) and the relations

$$\sigma_x = \frac{p}{m - 1} \quad \text{and} \quad \sigma_z = p,$$

the following equations may be derived.

$$\left.\begin{aligned} n = \sigma_r &= \frac{p}{m - 1} \sin^2\phi + p\cos^2\phi \\ \tau &= \left(p - \frac{p}{m - 1}\right) \sin\phi \cos\phi \end{aligned}\right\} . \qquad (9)$$

After introducing twice the angle ϕ and considering the direction of the stresses σ_t and τ the following equations may be easily obtained.

$$\left.\begin{aligned} \sigma_r &= \frac{p}{2(m - 1)} [m + (m - 2) \cos 2\phi] \\ \sigma_t &= \frac{p}{2(m - 1)} [m - (m - 2) \cos 2\phi] \\ \tau &= - \frac{p(m - 2)}{2(m - 1)} \sin 2\phi \end{aligned}\right\} . \qquad (10)$$

The above equations express the value of the stresses in any chosen direction.

If both principal stresses are of the same value, equation (9) takes the form

$$\sigma_r = \sigma_t = p, \quad \tau = 0 \cdot \qquad (11)$$

This means that in any chosen direction the value of the principal stress equals p and the shearing stress equals zero.

It may now be concluded that if the stresses on a cubic rock are equal in all directions, it cannot be destroyed since no shearing stresses exist.

Stress Conditions in a Plastic Medium

In order to generalize the foregoing equations correctly for a plastic medium, it is helpful to have them expressed diagrammatically with the aid of Mohr's circle(3).

Expressing σ_1 and σ_2 (both principal stresses) in the form

$$\frac{\sigma_1}{\sigma_2} = m - 1 \tag{12}$$

and with σ_n and τ (the normal and shearing stresses) forming angle ϕ with o z axis, (Fig. 2), one can, with the aid of Fig. 4, develop the following equations

$$\begin{aligned} \sigma_n &= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\phi \\ \tau &= \frac{\sigma_1 - \sigma_2}{2} \sin 2\phi \end{aligned} \tag{13}$$

These equations may be compared to equations (9) when p and $\frac{p}{m-1}$ have the values σ_1 and σ_2 and if the function of the double angle is introduced instead of ϕ. Thus the abscissa OB represents the principal stress and the ordinate BC the shearing stress.

The angle BAC = ϕ shows the direction of the stress σ with respect to direction OZ, or in other words with respect to the direction of the larger principal stress. From Fig. 4 it may also be seen that the largest value for the shearing stress occurs at an angle of 45° to the principal stress. This is shown by the second formula in group 13. The largest value of the quotient

$$\tan\rho = \frac{\tau}{\sigma_n} \tag{14}$$

may be located at the moving point p on the line OP. For a given value of α the following relationship holds true:

$$\sin\rho = \frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2}$$

or

$$\frac{\sigma_1}{\sigma_2} = \frac{1 + \sin\rho}{1 - \sin\rho} \quad . \tag{15}$$

If an imaginary plane cuts a loose sandy mass of earth and a considerable force is applied to the bordering parts causing a displacement along the plane, a certain frictional force will arise at the plane of contact. There will also be a slight movement of mass which is directly proportional to the force applied on the slide plane at the instant at which first motion takes place.

Taking μ as the proportionality constant between the applied force, σ_n and the frictional force on the imaginary plane or τ_n (n indicates the direction of the normal force on the observed particle) as the shearing stress, then the limiting formula may be given as

$$\tau_n = \mu\sigma_n \quad . \tag{16}$$

Now taking ρ as the frictional angle, formulae (14) and (16) agree when

$$\mu = \tan\rho \tag{17}$$

and formula (15) represents the highest value of the quotient of the two principal stresses.

Thus at any chosen point in the interior of a sandy mass the stress conditions will be as indicated by formula (15). The value of the perpendicular stress depends on the weight of the portion of earth above the point of observation, as expressed by formula (1). The horizontal principal stress will be expressed by formula (4) if the friction number is indefinitely large. If the friction number is finite, then the following formula holds true:

$$\frac{\sigma_1}{\sigma_2} = \frac{1 + \sin\rho}{1 - \sin\rho} \lesseqgtr m - 1 \quad . \tag{18}$$

If the upper sign is valid for a certain case then τ has a larger value than stipulated in formula (16). The single particle must therefore move along the slide planes which make an angle of $\frac{\pi}{2} - \rho$ between each other (angle ODP on the Mohr's circle) until the value given by formula (15) is reached. If the lower sign is valid then the value of τ is not certain and equation (16) changes to

$$\tau_n < \mu\sigma_n \quad . \tag{19}$$

If the miner had only to deal with media which have a coefficient of friction between 20° (clay and loam soil) and 40° (sand and broken rocks) and a Poisson's number between 4 and 7 one comes to the conclusion that only in rare exceptions ($\rho > 37°$, $m < 5$) will the horizontal principal stress be determined by Poisson's number.

By letting formula (18) equal to K - 1, by which

$$K = \frac{2}{1 - \sin\phi} \tag{20}$$

then by placing this value into formula (10), one gets the following stress conditions in an undisturbed cohesionless ground:

$$\left.\begin{aligned} \sigma_r &= \frac{p}{2(K-1)}\left[K + (K-2)\cos 2\phi\right] \\ \sigma_t &= \frac{p}{2(K-1)}\left[K - (K-2)\cos 2\phi\right] \\ \tau &= -\frac{p}{2}\frac{K-2}{K-1}\sin 2\phi \end{aligned}\right\} \tag{21}$$

(p = vertical principal stress).

If the value of the friction angle μ is introduced into formula (18), then the following formula may be obtained for the quotient of the principal stresses:

$$\frac{\sigma_1}{\sigma_2} = \left(\mu + \sqrt{1 + \mu^2}\right)^2 . \tag{22}$$

A miner rarely deals with non-cohesive ground but more frequently with rocks which have considerable cohesion, with the exception of clay, potter's clay and sand. Therefore shearing stresses which are produced by uniaxial stress conditions are opposed and may not be neglected.

In the general case the circumference of the stress circle (Fig. 5) will depend on, as Prandtl[4] emphasized, the curve $\tau = f(\sigma)$ which depends on the properties of the medium.

For the media which affect the miner (hard plutonic, igneous and sedimentary rock), experience shows that the shearing increases with the compressional stress and a straight line may be taken as the envelope to give sufficient accuracy.

For this case, where line OA = C (Fig. 6) represents the limiting flow in a plane stress condition (one of the three stresses equals zero), the following conditions may be set up:

$$\tau_n = c + \sigma_n \tan\rho \tag{23}$$

$$\frac{\sigma_1 - \sigma_2}{2} = (\sigma_o + c_1) \sin\rho \tag{24}$$

On the other hand the following conditions exist between the principal stresses at the limiting case

$$\frac{\sigma_1 - \sigma_2}{2} = \frac{\sigma_1 + \sigma_2}{2} \cdot \sin\rho + c_1 \sin\rho \cdot \tag{25}$$

Introducing the following value for the purpose of simplifying equations

$$\sigma = \sigma_o + c_1 \tag{26}$$

one gets

$$\frac{\sigma_1 + \sigma_2}{2} = \sigma - c_1 \ .$$

These relations enable one to set up reasonable simple formulae for the stress conditions in an undisturbed area at a large depth. They are:

$$\left.\begin{aligned} \sigma_r &= \sigma(1 + \sin\rho \cos 2\phi) - c_1 \\ \sigma_t &= \sigma(1 - \sin\rho \cos 2\phi) - c_1 \\ \tau &= \sigma\sin\rho \sin 2\phi \quad . \end{aligned}\right\} \tag{27}$$

In disturbed ground (the larger principal stress has any chosen direction with respect to OZ) the following are valid:

$$\left.\begin{aligned} \sigma_z &= \sigma(1 + \sin\rho \cos 2\phi) - c_1 \\ \sigma_x &= \sigma(1 - \sin\rho \cos 2\phi) - c_1 \\ \tau &= \sigma\sin\rho \sin 2\phi \quad . \end{aligned}\right\} \tag{28}$$

In large depths the simple equation (23) may be used to represent the stress conditions since c_1, the cohesion number, is small compared to σ.

To locate the range in which the formulae (10), (21), and (27) are valid, the following examples may be observed:

1. Plutonic, igneous and hard rock (granite, basalt, gneiss, etc):

$$\tau \sim 200 \text{ kg./cm}^2, \ \gamma \sim 2.7, \ m \sim 5, \ \mu \sim 0.75.$$

From the third formula in group (10) the depth at which the ultimate shearing stress occurs may be found ($\phi = 45°$)

$$200{,}000 = 2.7 \text{ h} \cdot \frac{3}{8} = \frac{8.1}{8} \text{ h}.$$

Therefore h = 1975 m. to approximately 2000 m. It can readily be seen that for average mining depths, equation (10) only may be used for the types of rocks mentioned above. Thus, using equation (10), the entire calculations must be solved with the aid of the theory of elasticity.

2. Medium-strong sandstone:

$$\tau \sim 40 \text{ kg./cm}^2, \ \gamma \sim 2, \ m \sim 5, \ \mu \sim 0.7.$$

At a depth of 533 m. the ultimate shearing strength will be reached by a plane inclined 45° to the horizontal. In this case the sandstone rock is found in a partially plastic condition since the friction resistance at this plane ($\phi = 45°$, see third formula, group (27)) is equal to 68 kg./cm.^{2} ($\sigma = 66.7 + c_1 = 120$ kg./cm.^{2}, $\tau = 120 \times 0.57 = 68$ kg./cm.^{2}) as a result no sliding can take place.

To determine the depth in a case in which equation (27) only is valid, and for a certain coefficient of friction we assume the slide plane to be at an angle of 27°30' $\left(\frac{\pi}{4} - \frac{\rho}{2}\right)$ with the vertical. Then the friction resistance for the above-mentioned depth is equal to

$$\tau = 120 \times 0.57 \times 0.82 = 56 \text{ kg./cm}^2$$

With the accepted numbers the stress condition given by formula (27) will not be reached until a depth of 1200 m. is arrived at.

If a disturbance (shafts or tunnels) occurs, then one of the two principal stresses will equal zero (the third has no influence). At this point the flow limit of the rock drops to 40 kg./cm.^{2} Under these conditions formula (27) becomes valid at a depth of 533 m.

It may be concluded that in the calculation of stress conditions in rock of medium hardness the flow zone may be located at a depth between 500 and 2000 m. depending on the smallest principal stress, shearing stress, Poisson's number and the coefficient of friction.

3. Clay, Loam and Marl:

$$\tau \sim 1 \text{ kg./cm.}^2, \ \mu = 0.36, \ m \sim 5, \ \gamma \sim 2.$$

In dealing with this type of rock the flow limit is reached at a depth of a few meters, thus formulae (21) only can be used satisfactorily at mining depths.

Stress Conditions in Disturbed Elastic Media

Stress Conditions Around an Opening of Circular Cross-Section

It is assumed that the horizontal opening is circular and the ground boundaries around it are infinite. A disc-shaped section perpendicular to the horizontal axis of the tunnel will be taken out of this opening at a point where its ends are far from the disc.

It is evident that the stress components which can be calculated from equations (10) and (27) undergo a basic change in the vicinity of the tunnel since the rock has a tendency to fill the tunnel immediately after boring takes place.

Because the displacement in the direction of the tunnel axis will be prevented by the adjacent mass of ground, according to the theory of elasticity[6], one can solve the problem by the use of the disc formula.

In rectangular coordinates these formulae are:

$$\frac{\partial^4 F}{\partial x^4} + 2\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = 0 \qquad (29)$$

$$\sigma_x = \frac{\partial^2 F}{\partial y^2}; \quad \sigma_y = \frac{\partial^2 F}{\partial x^2}; \quad \tau = c - \frac{\partial^2 F}{\partial x \partial y}. \qquad (30)$$

These equations in polar form (Fig. 7) or better expressed in cylindrical coordinates, since the stress in direction O Z is not zero, are:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2} + \frac{\partial^2}{\partial \phi^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2 F}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \phi^2} + \frac{1}{r}\frac{\partial F}{\partial r}\right) = 0 \qquad (31)$$

$$\left.\begin{aligned} \sigma_r &= \frac{1}{r^2}\frac{\partial^2 F}{\partial \phi^2} + \frac{1}{r}\frac{\partial F}{\partial r} \;;\; \sigma_t = \frac{\partial^2 F}{\partial r^2} \\ \sigma_z &= \frac{1}{m}(\sigma_r + \sigma_t); \quad \tau = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial \phi}\right) \end{aligned}\right\} \qquad (32)$$

The function F, which is known as Airy's stress function, was first introduced by Föppl for use in problems dealing with circular holes*. This function has a value as follows, when the principal stresses at infinity have the values p and $\dfrac{p}{m-1}$,

$$F = \frac{p}{4}\frac{m}{m-1}(r^2 - 2a^2 \log r) - \frac{p}{4}\frac{m-2}{m-1}\frac{(r^2 - a^2)^2}{r^2}\cos 2\phi \qquad (33)$$

From equation (32) one gets the values of the stress components which, with the exception of σ_z (the value of the principal stress in direction O Z) are shown in Fig. 3.

$$\left.\begin{aligned} \sigma_r &= \frac{p}{2}\frac{m}{m-1}\frac{r^2 - a^2}{r^2} \\ &\quad + \frac{p}{2}\frac{m-2}{m-1}\left(1 - \frac{4a^2}{r^2} + \frac{3a^4}{r^4}\right)\cos 2\phi \\ \sigma_t &= \frac{p}{2}\frac{m}{m-1}\frac{r^2 + a^2}{r^2} - \frac{p}{2}\frac{m-2}{m-1}\left(1 + \frac{3a^4}{r^4}\right)\cos 2\phi \\ \tau &= \frac{p}{2}\frac{m-2}{m-1}\left(-1 - \frac{2a^2}{r^2} + \frac{3a^4}{r^4}\right)\sin 2\phi \\ \sigma_z &= \frac{p}{m-1} - \frac{p}{m}\frac{m-2}{m-1}\frac{2a^2}{r^2}\cos 2\phi \;. \end{aligned}\right\} (34)$$

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* Obtained by superposition from the stress function of Kirsch, Z. VDI 42 (1898) p. 797.

By differentiation one can prove that equation (33) fulfills the conditions expressed in equation (31) and that equation (32) satisfies the boundary conditions for r = a,

$$\sigma_r = 0 \ ; \ \tau = 0,$$

and for

$$r \to \infty :$$

$$\sigma_r = \frac{p}{2(m-1)} [m + (m-2) \cos 2\phi]$$

$$\sigma_t = \frac{p}{2(m-1)} [m - (m-2) \cos 2\phi]$$

$$\tau = -\frac{p}{2} \frac{m-2}{m-1} \sin 2\phi$$

is also satisfied.

With the help of the known equation

$$\sigma_{\substack{max.\\ min.}} = \frac{\sigma_r + \sigma_t}{2} \pm \frac{1}{2} \sqrt{4\tau^2 + (\sigma_r - \sigma_t)^2} \tag{35}$$

one may put together, diagrammatically, the principal stresses around a circular opening as in Fig. 8.

With reference to Fig. 8, it is proven to the reader that the tension stresses which reach the highest value (0.25 p) at the edge of the circular opening, shatter the rock to considerable depth, so that fractures are formed on the sloping faces. For $\phi = n\pi$ the tension stresses come within the limits,

$$r = a \quad \text{and} \quad r = \pm \frac{a}{2} \sqrt{\pm \sqrt{m^2 + 24\ m = 48} - m}$$

so that there is, in reality, no value for the equation for shattered rock. On the other hand, the shearing stress has at

$$\phi = \frac{\pi}{4} + n\frac{\pi}{2} \quad \text{and } r = \pm a\sqrt{3}$$

the value

$$\tau_{max.} = -\frac{2}{3}\, p\, \frac{m-2}{m-1} \, .$$

Therefore, it is larger than the largest shearing stress at infinity.

This observation leads one to the conclusion that the developed formulae are of practical value only in media which can withstand high tension stresses and are not effective for media which have lost the property to withstand the tension stresses due to tectonic forces.

It follows from Kühn's work[6] that the stress conditions which occur around the face of the opening will cause small cracks in the rock. These cracks will relieve the ground from stresses, but stresses will occur in more remote zones resulting in new cracks. This condition is repeated and theoretically there is no end to this sequence. Since actual observations do not agree with the above statement, Kühn refers to the time factor which is not contained in the equations, and comes to the conclusion that there actually must be an equilibrium point.

It stands to reason that this time factor has no role as long as the stress conditions remain in elastic media below the elastic limit and in plastic media below the critical frictional resistance, since the equations refer to non-flow conditions. However, as soon as the medium starts to flow there is a certain velocity of flow, thus the factor of time must come into consideration. Kühn stated that he developed this theory only to prove ".... that the development of the Willman-Kommerell theory of existing stress free zones, which gives the starting point to many explorations and promises to solve problems on ground pressures, is to be considered as a mistake". Remembering that long before Kühn's publications the theory of stress conditions on edges of openings in elastic ground was known[7], and remembering that the form of stress-free zones agreed closely to the form of mathematical analysis, one comes to the conclusion that this material, as Kühn referred to it, did not start by the predecessors who worked on the problem. In contrast to Kühn's statement that "For forms and extension of the fractured zones, a theoretical mathematical formula can hardly be set up", and it will be proved that, to get an error-free result, it is sufficient to investigate the stress conditions around an opening with different cross sections and in that way the cross section which corresponds to the one demanded by nature may be found.

Stress Conditions Around an Opening with an Elliptical Cross-Section

The problem will be started in undisturbed ground, whose stresses are given by equations (10). An elliptical tunnel is driven in such a way that the main axis of the elliptical cross section coincides with the main axis of the stress-ellipse, so that the quotients of both axes of the ellipses have the same value of $m - 1$. In this case, for the computation of the stress components, it is natural to take elliptical coordinates, because of the limiting conditions:

$$y + ix = c \cosh (\xi + i\eta) . \tag{36}$$

By the law of complex numbers, one gets

$$\begin{aligned} y &= c \cosh \xi \cos \eta \\ x &= c \sinh \xi \sin \eta \end{aligned} \quad . \tag{37}$$

By substituting σ_ξ and σ_η for σ_r and σ_t into the formulae (29), (30) and (31)[8] and considering that for this special condition

$$h = \frac{c}{\sqrt{2}} \sqrt{\cosh 2\xi - \cos 2\eta} \tag{38}$$

one gets the following values,

$$\Delta\Delta F = \frac{4}{c^4 (\cosh 2\xi - \cos 2\eta)^3} \Bigg[(\cosh 2\xi - \cos 2\eta) \left(\frac{\partial^4 F}{\partial \xi^4} + \frac{2\partial^4 F}{\partial \xi^2 \partial \eta^2} + \frac{\partial^4 F}{\partial \eta^4} \right) - 4 \left\{ \sinh 2\xi \left(\frac{\partial^3 F}{\partial \xi^3} + \frac{\partial^3 F}{\partial \xi \partial \eta^2} \right) + \sin 2\eta \left(\frac{\partial^3 F}{\partial \xi^2 \partial \eta} + \frac{\partial^3 F}{\partial \eta^3} \right) \right\} + 4 (\cosh 2\xi + \cos 2\eta) \left(\frac{\partial^2 F}{\partial \xi^2} + \frac{\partial^2 F}{\partial \eta^2} \right) \Bigg] = 0 \quad . \tag{39}$$

$$\left.\begin{aligned} \sigma_\xi &= \frac{1}{h^2} \frac{\partial^2 F}{\partial \eta^2} + \frac{1}{h^3} \frac{\partial h}{\partial \xi} \frac{\partial F}{\partial \xi} - \frac{1}{h^3} \frac{\partial h}{\partial \eta} \frac{\partial F}{\partial \eta} \\ \sigma_\eta &= \frac{1}{h^2} \frac{\partial^2 F}{\partial \xi^2} - \frac{1}{h^3} \frac{\partial h}{\partial \xi} \frac{\partial F}{\partial \xi} + \frac{1}{h^3} \frac{\partial h}{\partial \eta} \frac{\partial F}{\partial \eta} \\ \tau &= - \frac{1}{h^2} \frac{\partial^2 F}{\partial \xi \partial \eta} + \frac{1}{h^3} \frac{\partial h}{\partial \eta} \frac{\partial F}{\partial \xi} + \frac{1}{h^3} \frac{\partial h}{\partial \xi} \frac{\partial F}{\partial \eta} \\ \sigma_z &= \frac{1}{m} (\sigma_\xi + \sigma_\eta) \quad . \end{aligned}\right\} \tag{40}$$

The Airy's function stated by Pöschl*, has the following value in a case where the principal stress is inclined at any chosen angle $\left(\alpha + \frac{\pi}{2}\right)$. See Fig. 9.

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* The same solution by K. Wolf appeared at the same time, Z. techn. Physik 2, 1921. p. 209.

$$F = \frac{pc^2}{8} \Big\{ \sinh 2\xi - \cos 2\,\alpha e^{-2(\xi-\xi_o)} -$$
$$- 2(\cosh 2\xi_o + \cos 2\alpha)\xi +$$
$$+ [\cosh 2(\xi - \xi_o) - 1]\, e^{2\xi_o} \cos 2(\eta - \alpha) \Big\}. \qquad (41)$$

By superposition, one is able to calculate Airy's stress function for the general case (equation (10)) for a chosen inclination of the main axis of both ellipses.

$$F = \frac{pc^2}{8(m-1)} \Big\{ m \sinh 2\xi + (m-2) \cos 2\alpha e^{-2(\xi-\xi_o)} -$$
$$- 2\,[m \cosh 2\xi_o - (m-2)\cos 2\alpha]\xi -$$
$$- (m-2)\,[\cosh 2(\xi - \xi_o) - 1]\, e^{2\xi_o} \cos 2(\eta - \alpha) \Big\}. \qquad (42)$$

For the special case, one must put

$$\alpha = 0 \quad \text{and} \quad \tanh \xi_o = \frac{b}{a} = \frac{1}{m-1}$$

thus giving equation (41) the form

$$F = \frac{pc^2}{8(m-1)(m-2)} \Big\{ m(m-2) \cosh 2\xi -$$
$$- 4(m-1)\xi - [m(m-2) + 2] \cosh 2\xi \cos 2\eta +$$
$$+ 2(m-1) \sinh 2\xi \cos 2\eta + m(m-2) \cos 2\eta \Big\}. \qquad (43)$$

By differentiation, it can be proved that equation (43) satisfies the conditions set up in equation (39). The stress components attain the following values:

$$\left.\begin{aligned} \sigma_\xi &= \frac{mp}{2(m-1)} - \\ &- p\,\frac{m^2(1 - \cosh 2\xi \cos 2\eta) + 2(m-1)(e^{2\xi} \cos 2\eta - 1)}{2(m-1)(m-2)(\cosh 2\xi - \cos 2\eta)} \\ \sigma_\eta &= \frac{mp}{2(m-1)} + \quad \text{(cont'd. next page)} \end{aligned}\right\} (44)$$

$$+ p \frac{m^2(1 - \cosh 2\xi \cos 2\eta) + 2(m-1)(e^{2\xi} \cos 2\eta - 1)}{2(m-1)(m-2)(\cosh 2\xi - \cos 2\eta)}$$

$$\tau = p \frac{2(m-1)\, e^{2\xi} - m^2 \sinh 2\xi}{2(m-1)(m-2)\, \cosh 2\xi - \cos 2\eta} \sin 2\eta$$

$$\sigma_z = \frac{p}{m-1}$$

Now it may be stated that the stress components fulfil the following conditions:

$$\left.\begin{aligned} \sigma_{\xi_{\xi=\xi_0}} &= 0, \quad \tau_{\xi=\xi_0} = 0. \\ \sigma_{\xi_{\xi \to \infty}} &= \frac{p}{2(m-1)} [m + (m-2) \cos 2\eta] \\ \sigma_{\eta_{\xi \to \infty}} &= \frac{p}{2(m-1)} [m - (m-2) \cos 2\eta] \end{aligned}\right\} \qquad (45)$$

$$\tau_{\xi \to \infty} = - \frac{p}{2} \frac{m-2}{m-1} \sin 2\eta . \qquad (46)$$

This proves that the solution is correct.

In Figs. 10, 11, and 12 the values of the stresses are diagrammatically shown for m = 5 and b:a = m - 1. One can see that the tension stresses have disappeared and the value of the tangential normal stress on the edge of the elliptical opening is constant and is:

$$\sigma_{\eta_{\xi \to \xi_0}} = \frac{mp}{m-1} . \qquad (47)$$

Therefore, this value is equal to the sum of the principal stresses produced in infinity. The sum remains constant for any point outside the opening as can be seen from the first two equations of group 44.

That the found cross section gives the most favourable from all possible stress distributions, can be seen from the circumstances that the sum of the three stress components stays constant and is equal to the sum of the three principal stresses at infinity. Also the largest value of the shearing stress at infinity decreases directly proportional to the distance and therefore equals zero at the edge of the opening.

The high tangential pressure in the suspended part of the tunnel holds the blocks of rock in broken ground due to its friction. Also in the case of perfectly smooth sliding faces a coefficient of friction larger than 0.3 can safely be taken, so that for $m = 5$ the friction resistance is not smaller than

$$R = 0.3 \times \frac{5}{4} \times p$$

or for depths of more than 100 m. will not be smaller than 8 kg./cm.^{2} It can then be calculated that such a block, if rock, would have to have dimensions which would far surpass the width of the largest tunnel. (For a sp. gr. of 2.5 the sliding planes, which are parallel to the tunnel axis, would have to be more than 60 m. apart before the block could fall down.)

This condition proves that the accepted cross section will never be reached in nature since the so-called tangential normal stress will prevent the downfall of the rock block, and therefore the final equilibrium condition is restored long before the occurrence of these cross sections.

To find the natural cross section, one starts with the following facts: the circular cross sections cause tension stresses which break the rock resulting in a stress-free zone: the ellipse under investigation causes, on the other hand, compression stresses which prevent the formation of these fractures. Naturally, there must exist a form where the stresses at the crown are equal to zero, namely for an ellipse in which

$$\sigma_\xi = \sigma_\eta = \tau = \sigma_z = 0 \quad . \tag{48}$$

To find this ellipse, the value $\alpha = 0$ is introduced into equation (42) and the following value is obtained:

$$F = \frac{pc^2}{8(m-1)} \{m \sinh 2\xi + (m-2)\, e^{-2(\xi-\xi_0)} -$$

$$- 2\,[m \cosh 2\xi_0 - m + 2]\xi -$$

$$- (m-2)\,[\cosh 2(\xi - \xi_0) - 1]\, e^{2\xi_0} \cos 2\eta\} \, . \tag{49}$$

The second formula in group (40) gives the value

$$\sigma_\eta = \frac{p}{m-1}\, \frac{m \sinh 2\xi_0 + (m-2)(1 - e^{2\xi_0} \cos 2\eta)}{\cosh 2\xi_0 - \cos 2\eta} \, . \tag{50}$$

Taking then $\eta = 0$, the value of the stress equals zero and one gets the following equation:

$$\tanh\xi_0 = \frac{b}{a} = \frac{2}{m - 2} \tag{51}$$

and the Airy's stress function takes the form:

$$\begin{aligned} F = \frac{pc^2}{8(m-1)(m-4)^2} \{&m(m-2)(m-4) \cosh 2\xi - \\ &- 2m(m-4) \sinh 2\xi - 4m(m-4)\xi - (m-2) \\ &[(m-2)^2 + 4] \cosh 2\xi \cos 2\eta + 4(m-2)^2 \\ &\sinh 2\xi \cos 2\eta + m(m-2)(m-4) \cos 2\eta\} \quad , \end{aligned} \tag{52}$$

while the stress components have the following equations:

$$\left.\begin{aligned} \sigma_\xi &= \frac{m(m-2)}{2(m-1)(m-4)} p + p \frac{m(m-4)(1 - \cosh 2\xi) + 8}{(m-1)(m-4)^2(\cosh 2\xi - \cos 2\eta)^2} \sinh 2\xi + \\ &+ p \frac{(m-2)\,[(m-2)^2 + 4]\,(\cosh 2\xi \cos 2\eta - 1) - 4(m-2)^2 \sinh 2\xi \cos 2\eta}{2(m-1)(m-4)^2\,(\cosh 2\xi - \cos 2\eta)} \\ \sigma_\eta &= \frac{m(m-2)}{2(m-1)(m-4)} p - p \frac{m(m-4)(1-\cos 2\eta) + 8}{(m-1)(m-4)^2\,(\cosh 2\xi - \cos 2\eta)^2} \sinh 2\xi - \\ &- p \frac{(m-2)[(m-2)^2+4](\cosh 2\xi \cos 2\eta - 1) - 4(m-2)^2 \sinh 2\xi \cos 2\eta + 2m(m-4)\sinh 2\xi}{2(m-1)(m-4)^2\,(\cosh 2\xi - \cos 2\eta)} \\ \tau &= (m-2)\, p \frac{4(m-2)\, e^{2\xi} - m^2 \sinh 2\xi}{2(m-1)(m-4)^2(\cosh 2\xi - \cos 2\eta)} \sin 2\eta + \\ &+ p \frac{m(m-4)(1 - \cosh 2\xi) + 8}{(m-1)(m-4)^2(\cosh 2\xi - \cos 2\eta)^2} \sin 2\eta \\ \sigma_z &= \frac{m-2}{(m-1)(m-4)} p - p \frac{2 \sinh 2\xi}{(m-1)(m-4)(\cosh 2\xi - \cos 2\eta)} \; . \end{aligned}\right\} \tag{53}$$

By substituting the corresponding values, it can be proved that these equations completely satisfy the conditions set up.

From the conditions set up it can be seen that every opening eventually becomes elliptical in form (except when $m = 4$) and that tension stresses occur, within the ellipse set up by equation (51), which breaks the rock and completely frees it of stresses, and with that it is mathematically proven to be a stress-free zone (by Spackeler). In Figs. 13 and 14 the stress values for m = 5 and m = 7 are shown. It can be seen that the elliptical opening has an influence on its surroundings, the influence depending on the dimensions of the opening. Though the formulae give the primary stresses only at infinity, in reality the values have high gradients in the vicinity of the opening so that at a distance of 4a to 5a the stress values can hardly be distinguished from the primary values.

This proves that the results obtained must correspond very closely to the natural conditions, even though the specific gravity of the rock is neglected, since the accuracy of the formulae depend on the relations between the depth of the tunnel and the radius of the influenced zone.

Also, if the natural, observed conditions should not agree completely with the calculations since the rock is neither homogeneous nor does it strictly obey Hooke's law, it should still be remembered that a stressless zone does exist around the opening, stresses being taken up by the surrounding ground. Thus, it can be said that the investigation has been completed satisfactorily.

One must bear in mind that at larger depths the stresses at the boundary of an opening may be larger than the breaking strength of the rock concerned, since

$$\sigma_{\eta\,\max.} = \frac{m^2}{(m-1)(m-2)}\,p$$

and therefore for m = 5, this value of σ_η = 2.083 p, so that the rock might be crushed and thus cause an enlargement in the width as well as the height. In this case equilibrium will never be reached. In the following section the answer may be obtained to this open question.

Stress Conditions around an Opening with a Circular Cross Section in a Plastic Medium

Vertical Shaft

With reference to Fig. 19, assume a layer PQ at depth h which surrounds a shaft and extends to infinity. Also assume that horizontal slits have been left inside the shaft to allow the medium to flow freely thus relieving the pressures. Furthermore, a yielding shaft may be taken into consideration thus allowing the masses around the shaft which are under high stresses to flow freely towards the shaft. The shaft may be considered as made in wet pure clay which is known to be very plastic, furthermore it is assumed that the entire shaft contains slits so that many tons of material may be removed from the shaft daily. Thus the question arises whether in clay with $\mu = 0.36$ and $c = 1$ kg./cm.^{2}, it is possible to have equilibrium at a depth of 1000 m. so that flow of masses is made impossible.

Since the cohesion for this unfavourable case is set to be 1 kg./cm.^{2}, one can calculate the primary stress conditions with the aid of formula (21), where σ_r, for $\phi = 0$, equals p or if $\gamma = 2.4$ and h = 1000 m. amounts to 240 kg./cm.^{2} σ_t represents the horizontal principal stress in any direction when $\phi = 0$, when a rotational stress ellipsoid is considered then the following equation is valid:

$$\sigma_t = \frac{240}{K - 1} = \frac{240}{(\mu + \sqrt{1 + \mu^2})^2} = 120 \text{ kg./cm.}^2$$

Observing the layer itself, it can be seen that the stresses are everywhere the same (the weight of the layer can be neglected since it can be chosen as thin as we please), therefore the stress ellipse is a circle.

If we imagine a coordinate system in the layer where the x-y plane is the plane of symmetry in the layer, then the primary stress conditions can be expressed by the formula $\sigma_x = \sigma_r = 120$ kg./cm.^{2} $\sigma_y = \sigma_t = 120$ kg./cm.^{2}, $\tau = 0$ (in this case the third principal stress is the bigger one). On the basis of symmetry, it is convenient to introduce polar coordinates where the origin coincides with the centre of the circular shaft.

After sinking the shaft the entire surroundings start to move, and move in the direction of the radius on which they were at the start of the movements (considering a circular cross section). On the other hand, all grains on the same circle before boring move the same amount so that between two adjoining grains there is no relative displacement and therefore no shearing stresses occur since the shearing stress before movement was equal to zero. The direction of the radius and tangent will also be in the principal direction after movement, and therefore the principal stresses must coincide with the direction of the radius and tangent. This condition aids in the establishment of the equation of the flow lines since the flow lines cut the principal stresses at a constant angle and therefore also the direction of any radius.

As known, this condition is fulfilled by the logarithmical spiral whose equation is $r = ae^{m\phi}$, since the tangent at a point P forms a constant angle $\psi = \tan^{-2}m$ with the radius vector. But since ψ can have two values

$$\psi' = \frac{\pi}{4} + \frac{\rho}{2} \quad \text{and} \quad \psi'' = \frac{\pi}{4} - \frac{\rho}{2},$$

the sheaf of slip lines is still not determined.

To locate these lines one must establish the direction of the larger principal stress, which can coincide with the direction of a radius as well as with the corresponding tangent of any one of the circles concentric with the shaft. Keeping in mind that after sinking a shaft, σ_r suddenly becomes very small and at the same time σ_t tends to get larger as the circumference of the circle becomes smaller due to the movement of the grains, one comes to the conclusion that the equilibrium between principal stresses can be obtained only when the equation

$$(K - 1)\sigma_R = \sigma_T \tag{54}$$

holds true. This equation gives the biggest possible value that σ_T can have for a given value of σ_R. The capital letters for subscripts indicate principal stresses*.

Alternatively, σ_R can never be smaller than σ_T when obtained by equation (54) because it would cause a run-off of the medium in the opening. This would upset the basic principles of the argument. Here one must try to represent the stress conditions when at rest, and so find under what conditions the ground will be in equilibrium.

This calculation has been made dependent upon the determination of σ_R because if this value is known one can find the direction and magnitude of all the stresses.

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* The other condition of principal stresses is only possible in a medium which possesses tensile strength. Expansion of a hole by means of a bolt (see flow figures in Hütte: Mechanik der bildsamen Körper, vol. 1.)

To represent the differential equation, one needs to know only that in this special case the formulae (32) are independent of ϕ for symmetrical reasons, thus one has the following values

$$\sigma_R = \frac{1}{r}\frac{\partial F}{\partial r} \; ; \quad \sigma_T = \frac{\partial^2 F}{\partial r^2} \; ; \quad \tau = 0 \; . \tag{55}$$

Taking the condition (54) into consideration, one gets

$$\frac{\partial^2 F}{\partial r^2} - (K - 1)\frac{1}{r}\frac{\partial F}{\partial r} = 0 \; . \tag{56}$$

By solving this differential equation one gets

$$F = \frac{C}{K} r^K \tag{57}$$

and obtains*

$$\sigma_R = Cr^{K-2} \; ; \quad \sigma_T = C(K-1)\, r^{K-2} \; .$$

To find the constant C, one takes σ_{R_a} as the stress on the shaft itself. One puts $r = a$ and gets $C = \frac{\sigma_{R_a}}{a^{K-2}}$.
The stresses than have the values

$$\sigma_R = \sigma_{R_a}\left(\frac{r}{a}\right)^{K-2} \; ; \quad \sigma_T = (K - 1)\,\sigma_{R_a}\left(\frac{r}{a}\right)^{K-2} \; . \tag{58}$$

For the accepted value of $\mu = 0.36$, or more exactly $\mu = 0.35354$,

$$\sigma_R = \sigma_{R_a}\,\frac{r}{a} \; ; \quad \sigma_T = 2\,\sigma_{R_a} \times \frac{r}{a} \tag{59}$$

i.e., the stress increases in direct proportion as the distance from the shaft.

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* The same problem was solved by Hartmann in a different way. See Nadai, Handbuch der Physik, vol. 6: Das Gleichgewicht locherer Massen.

Taking into consideration that the lowest friction angle for mining media is 20° and thus $\mu = 0.364$, it is evident that for some mining depth equilibrium will be established around the shaft and the medium must come to rest. Therefore, at some depth the pressure on the shaft can drop to the lowest cohesion values. This was mentioned for the reason that it would be hard to explain the openings in the ground made by small animals in ground with poor cohesion properties. If a poorly stressed zone would not appear around a shaft in time there would be no explanation for the fact that after a certain length of time the openings in a mine come to rest.

To prove that the equations (58) are the only solution to the problem, it is necessary to determine the boundary conditions on the border of the flow zone and prove that equations (58) satisfy this condition. Since the medium consists of elastic grains Hooke's law holds as long as the quotient of principal stresses is kept below the critical values. Outside of the flow zone, the quotient will drop from its critical values (at the boundary of the flow zone) to the value one (at infinity) so that the formulae of the layer for this case are independent of ϕ and have the following values, since at infinity the principal stresses are equal to each other.

$$\left.\begin{aligned} &\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2 F}{\partial r^2} + \frac{\partial F}{\partial r}\right) = 0 \\ &\sigma_r = \frac{1}{r}\frac{\partial F}{\partial r}\ ; \quad \sigma_t = \frac{\partial^2 F}{\partial r^2}\ ; \ \tau = 0 \\ &\sigma_z = p_s - \frac{2\,p_s}{m(m-1)} + \frac{1}{m}(\sigma_r + \sigma_t)\ . \end{aligned}\right\} \qquad (60)$$

With the aid of Airy's stress function:

$$F = C_o + C_1 \log r + C_2 r^2 + C_3 r^3 \log r \qquad (61)$$

where in this case C_3 can be made equal to zero, since the angle at the centre remains constant outside the critical zone, after introducing the boundary conditions, we arrive at the following equations,

$$\left.\begin{aligned} \sigma_r &= p - (p - p')\,\frac{b^2}{r^2} \\ \sigma_t &= p + (p - p')\,\frac{b^2}{r^2} \end{aligned}\right\} \qquad (62)$$

where p is the compressive pressure at infinity, p' the compressive pressure at the boundary of the flow zone and b the radius of the flow zone. To compute the value of p' one assumes that the sum of the two principal stresses must be constant and equals:

$$\sigma_r + \sigma_t = 2\,p \quad . \tag{63}$$

On the boundary of the flow zone, σ_t reaches the largest possible value which, on the one hand is found by the use of equation (54), and on the other hand by equation (63), therefore, the equation

$$\sigma_R + \sigma_R(K - 1) = 2\,p$$

follows, from which σ_R can easily be computed;

$$p' = \sigma_R = \frac{2p}{K} \quad . \tag{64}$$

If one substitutes this value in equations (58) and also puts $r = b$, one gets

$$\frac{2p}{K} = \sigma_{R_a} \left(\frac{b}{a}\right)^{K-2} \tag{65}$$

and from that, the value b can be computed. When equilibrium is reached, the values of the principal stresses within the flow zone are as follows

$$\left.\begin{aligned} \sigma_R &= \sigma_{R_a} \left(\frac{r}{a}\right)^{K-2} \\ \sigma_T &= (K - 1)\,\sigma_{R_a} \left(\frac{r}{a}\right)^{K-2} \quad . \end{aligned}\right\} \tag{66}$$

Outside the flow zone, the following are the values of the stresses.

$$\left.\begin{aligned} \sigma_R &= p\left[1 - \frac{b^2}{r^2}\left(1 - \frac{2}{K}\right)\right] \\ \sigma_T &= p\left[1 + \frac{b^2}{r^2}\left(1 - \frac{2}{K}\right)\right] \quad . \end{aligned}\right\} \tag{67}$$

In Fig. 20 the values of the stresses for $\mu = 0.75$ (dotted lines) and $\mu = 0.5773$ (solid lines) are given on the top, and on the bottom they are given for $\mu = 0.35354$ where it was assumed, for $\sigma_{R_a} = 1, \gamma = 2.4$ and $h = 1000$ m. With these values, one gets the following values for the original horizontal pressure p, for a vertical value of 240 kg./cm^2 $(\gamma \times h)$:

Coeff. of friction μ	Quotient K - 1	Exponent K - 2	Flow zone b	Pressure p
0.75	4	3	2.8845 a	60
0.5773	3	2	6.3246 a	80
0.35354	2	1	80.0 a	120

The third column gives the value of the exponent so that the stresses are proportional to the cube or the square of the distance, respectively, for the first two coefficients of friction. For the last coefficient of friction the stress increases proportionately to the distance. As can be seen from Fig. 20, radius b of the flow zone increases from b = 2.8845 a at $\mu = 0.75$ to b = 80 a at $\mu = 0.35354$. This proves that in clay the computed stresses will be arrived at, after a very long time since the movement of the medium is slow due to the high coefficient of friction.

In practice when considering shafts that are concrete-lined and shafts built in media with a low coefficient of friction one can never predict the value of σ_{R_a}, since it depends mainly on the material removed during excavation. Therefore, one can gather that in media with a high coefficient of friction, equilibrium will come in a relatively short time and the effective pressure on the shaft will be very low and practically independent of the depth since the third principal stress σ_z must take the value

$$\sigma_r \leqq \sigma_z \leqq \sigma_t \qquad (68)$$

and therefore equilibrium exists.

In introducing the third principal stress, which has been so far considered constant in computing stresses outside the flow zone, and also introducing its own weight, in three dimensional computations one is forced to prove that the third principal stress can never be negative or smaller than σ_{Ra}, otherwise the stresses will not be in equilibrium. To give the proof from basic principles, Fig. 21 should be observed in which a spherical opening within a plastic medium is taken into consideration. Suppose that a concentric element is cut out of the material surrounding this opening at a radius r and vertically above it. Take σ as the radial stress, γ as the specific weight and $d\phi$ as the angle between the lines, dr as the thickness of the element, and a as the radius of the spherical opening.

The medium should have the freedom to expand and flow into the opening as in the preceding examples. For symmetrical reasons the sides of the element must be planes of principal stress so that no shearing stresses can occur. On the other hand, the principal stresses on the sides must have the limiting value $(K - 1)\sigma$ and that on the top surface equal $\sigma+(\partial\sigma/\partial r)dr$ (when the bottom stress is taken as σ).

Taking the vertical as the projection axis, one gets the following equation:

$$2(K - 1)\ \sigma \times r \times dr \times d\phi^2 - \gamma r^2\ dr\ d\phi^2 -$$
$$- \frac{\partial\sigma}{\partial r}\ r^2\ dr\ d\phi^2 - 2\sigma r\ dr\ d\phi^2 = 0.$$

(The infinitely small values of the fourth order were neglected.) Cancelling the common factors and dividing the equation by r, one obtains the differential equation:

$$\frac{d\sigma}{dr} - 2(K - 2)\ \frac{\sigma}{r} + \gamma = 0 \qquad (69)$$

the solution of which is as follows:

$$(K - 5)\sigma = C\ r^{2(K-2)} + \gamma r.$$

Substituting for $r = a$, $\sigma = \sigma_a$, one gets

$$C = \frac{2(K - 5)\ \sigma a - \gamma a}{a^{2(K-2)}}\ ,$$

and the equation becomes of the form:

$$\sigma = \left(\sigma_a - \frac{\gamma a}{2\,K - 5} \right) \left(\frac{r}{a}\right)^{2(K-2)} + \frac{\gamma r}{2\,K - 5} \quad . \qquad (70)$$

For $\gamma = 0$ this equation is effective for any direction and represents three-dimensional stress conditions. But if one takes the specific weight into consideration (i.e., $\gamma \neq 0$) then the equation is valid for the $0\,Z$ axis only.

Considering now the lowest value $K = 3$ one gets:

$$\sigma = (\sigma_a - \gamma a) \left(\frac{r}{a}\right)^2 + \gamma r \qquad (71)$$

therefore the equation can have no negative values where $\sigma_a > \gamma a$.

Taking $a = 400$ cm. and $\gamma = 2.5$ so that the first member for $\sigma_a = 1$ kg./cm.^{2} disappears and σ increases in proportion to the distance. The flow zone will extend to a third part of the observed depth measured from the origin of the coordinates since at that point both principal stresses reach the primary value.

For $K = 2.5$ one gets:

$$\sigma = \sigma_a \frac{r}{a} - \gamma r \log \frac{r}{a} \qquad (72)$$

because

$$\lim_{K \to 2.5} \frac{a\left(\frac{r}{a}\right)^{2(K-2)} - r}{2\,K - 5} = r \log \frac{r}{a} \quad .$$

On the surface $\sigma = 0$, therefore

$$\log \frac{r}{a} = \frac{\sigma_a}{a\gamma} \quad .$$

Because $r:a = 250$, σ_a must be equal to 5.52146 kg./cm.^{2} to prevent the flow of the medium at $a = 4$m. $r = 1000$ m. and $\mu = 0.204$. Since a friction angle of 11°30' does not come into consideration in practice, the above gives theoretical values only.

To compare the two examples, one considers that, neglecting the weight of the sphere in equation (71), the stress for K = 3 increases as follows:

$$\sigma = \sigma_a \left(\frac{r}{a}\right)^2 \qquad (73)$$

compared to the slice (equation (59)) for which the following formula is valid:

$$\sigma = \sigma_a \frac{r}{a} \quad . \qquad (74)$$

For shafts the latter equation is used (equation (74)), therefore equation (73) becomes invalid.

To prove that the vertical value of the principal stress is always positive, one considers that the vertical principal stress on the sphere (σ_s) is a function of the horizontal principal stress (σ_h) as follows:

$$\sigma_s = \sigma_h - \gamma a \sqrt{\frac{\sigma_h}{\sigma_a}} \left(\sqrt{\frac{\sigma_h}{\sigma_a}} - 1\right) \quad . \qquad (75)$$

In order that σ_s should not become negative, σ_h should be positive and the remainder of the equation is always less then σ_h. In the case of shafts, σ_h is found by the use of equation (74). Substituting this value in equation (75), one obtains

$$\sigma_s = r \left(\frac{\sigma_a}{a} - \gamma\right) + \gamma\sqrt{ar} \quad . \qquad (76)$$

To prevent any tension stresses in this case, it is necessary that $\sigma_a \geqq a\gamma$. The true value of σ_s must be between the extreme limits of equation (75) and (76), so that in equation (75) it may be considered that the third principal stress is inclined at $d\phi/2$ to the radius. Equation (76) is derived by taking the principal stress planes parallel to the considered radius. With that, it is proved that as long as $\sigma_a \geqq a\gamma$ it is impossible for tension stresses to appear, and the investigation can be considered as completed. It can be concluded that in order to keep a shaft open and safe in any medium, σ_a must be larger than γa so that equilibrium can be established.

To approximate the amount of material that must be taken out of the shaft to obtain equilibrium conditions, one takes the extended volume equal to:

$$e = (\sigma_x + \sigma_y + \sigma_z)\,\frac{m-2}{mE} \quad . \tag{77}$$

Taking $\gamma = 2.4$, $\mu = 0.35354$, $h = 1000$ m., $a = 3$m., $m = 5$ and $E = 300{,}000$ kg./cm.^{2}, then for the primary stress condition one gets $e = 0.00096$. Since the flow zone has a diameter of 480 m., for the total primary volume for each meter of shaft one gets $V_1 = 173$ m.^{3} After the stresses relax, the volume is equal to $V_2 = 96.3$ m.^{3}, if one takes σ_a as a unit and σ_z as the value of the largest principal stress.

The difference of 76.7 m.^{3} represents the amount of material that must flow into the shaft to reach equilibrium. By making the shaft deeper (a = 3m.) the amount of excavated material is about 28 m.^{3}, thus one can conclude that in general more than 100 m.^{3} of ground must be removed to reach final e q uilibrium.

According to K. Terzaghi[9], the water content of clay is a function of the pressure. In Fig. 22 the relationship between the pressure and water content of a unit volume of clay is shown diagrammatically. As can be seen from the above diagram, the water content decreases to one-half of the original value when the pressure increases from 0 to 8 atmospheres. On the other hand, if pressure is decreased clay will absorb the water. This explains the known fact that there is very little difference, between an opening driven through clay and one driven in clayey sandstone, since the water content in the clay is very low due to the high primary pressure and the clay shows a high cohesion. In a few days the opening starts to break and the amount of material which must be removed to keep the opening is even larger than that calculated above. This is because the opening not only depends on the pressure but also on the amount of moisture that is being absorbed.

This continues very slowly and the volume approaches the new equilibrium stress asymptotically. Thus it is obvious that the new stress condition will reach equilibrium after an infinitely long time. These two factors (time, and the material removed from the shaft) force the miners to line the shaft, and the civil engineer to line tunnels. In Simplon (largest tunnel in Europe) the tunnel on a 40 m. stretch has a lining of 2 m. average thickness.

In Fig. 23 (left) the effect of the pressure on the radius of the flow zone is shown diagrammatically for $\mu = 0.35354$, $a = 3$m., $\gamma = 2.4$ and $h = 1000$ m. In Fig. 23 (right) the volume for the values is given. It can be seen that the flow zone and volumes decrease very rapidly with the increase in pressure.

The above investigation discusses an unfavourable case. Now take $c = 15$ kg./cm.^{2}, $\mu = 0.5774$ (very soft clay sandstone) and one gets a value of 6 m. for the radius of the flow zone for $h = 1000$ m., $\gamma = 2.4$, $m = 5$, $a = 3$ m. and $E = 300{,}000$ kg./cm.^{2} The volume expansion is 28.5 m.^{3} per metre of depth. The last number corresponds to a decrease of the primary radius of the shaft by approximately 1 cm.

Before going to the next case, the relation of the flow zone to the depth should be shown. Taking the horizontal primary pressure as

$$p_h = \frac{\gamma \times h}{k - 1}$$

and using equation (65) one gets the following expression for the radius b of the flow zone:

$$b = a \sqrt[K-2]{\frac{2}{K(K-1)} \cdot \frac{\gamma h}{\sigma_{R_a}}} \quad . \qquad (78)$$

Naturally only values which are positive and equal to or larger than one are valid in the radical sign. In Fig.24 the flow zone is represented for $a = 4$m., $\gamma = 2.5$, $\sigma_{R_a} = 5$ kg./cm.^{2}, $\mu = 0.35354$, therefore $K = 3$ (dotted line), $\mu = 0.5773$, thus $K = 4$ (broken lines) and $\mu = 0.75$, thus $K = 5$ (solid line). It may be observed that the smallest increase of the flow zone occurs at the highest value of coefficient of friction.

Horizontal Tunnels or Adits

Here an investigation of horizontal openings driven for a distance h into a plastic, homogeneous medium will be made. As in the preceding examples, this opening shall have slits, or be yielding, which will allow the medium surrounding the adit to yield and become free of stresses. To bring the diagram of the stress conditions from space to a plane condition, one takes into consideration a layer normal to the tunnel axis which is a sufficient distance from the end points of the tunnel. One imagines the polar coordinates put in this cross section in such a way that the origin coincides with the projection of the tunnel axis, and that the angle ϕ be measured from the vertical.

From Fig. 25 one gets the equations:

$$\left.\begin{aligned} \sigma_r + r\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau}{\partial \phi} - \sigma_t &= -r\gamma\cos\phi \\ 2\tau + r\frac{\partial \tau}{\partial r} + \frac{\partial \sigma_t}{\partial \phi} &= r\gamma\sin\phi \end{aligned}\right\} \tag{79}$$

Introducing the stress function F, by differentiation, one can express the stresses as follows:

$$\left.\begin{aligned} \sigma_r &= \frac{1}{r^2}\frac{\partial^2 F}{\partial \phi^2} + \frac{1}{r}\frac{\partial F}{\partial r} - \frac{2}{3}r\gamma\cos\phi \\ \sigma_t &= \frac{\partial^2 F}{\partial r^2} \\ \tau &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial \phi}\right) + \frac{1}{3}r\gamma\sin\phi \end{aligned}\right\} \tag{80}$$

If the stresses satisfy the equation

$$\sigma_r + \sigma_t = \frac{K}{\sqrt{K-1}}\sqrt{\sigma_r \times \sigma_t - \tau^2} \tag{81}$$

and the boundary conditions, then the problem can be considered as solved.

Introducing the function

$$F = r^K \Sigma\, A_n \cos n\phi \tag{82}$$

where A_n represents the parameter which depends on the boundary conditions, one can find, in special cases (10) an approximate solution. For the observed case, it is difficult to determine the conditions for the boundaries of the flow zone as its form is unknown. Still one may assume that the outer limit of the flow zone has the shape of an ellipse as will be seen from the following explanations.

Taking A B Fig. 26 as a curve made of sand grains so that it cannot transmit any tension stresses but is able to withstand compression, furthermore taking T and T + dT as tangential forces which appear in the arc produced, P and P' as two fixed points on the curve A B, ds as an infinitely small part of the same curve, and finally σ_r and τ, as the produced stresses which are in this case given by equations (1) and (3) of group (21), one gets tangent and radius of the point P as the projection axes :

$$(T + dT) \cos d\phi - T + \tau ds = 0$$

$$(T + dT) \sin d\phi - \sigma_r ds = 0.$$

Substituting the values of τ and σ_r given by the formulae in these equations, and by neglecting the small values of the second order one gets the equations:

$$\left.\begin{aligned} dT &= \frac{p}{2} \frac{K - 2}{K - 1} ds \sin 2\phi \\ Td\phi &= \frac{p}{2(K - 1)} ds \left[K + (K - 2) \cos \phi\right] \end{aligned}\right\} . \qquad (83)$$

By eliminating ds one obtains

$$\frac{dT}{T} = \frac{1}{2} \quad \frac{-2(K - 2) \sin 2\phi \, d\phi}{K + (K - 2) \cos 2\phi} ,$$

and by integration one gets

$$T = \frac{C}{\sqrt{K + (K - 2) \cos 2\phi}} ,$$

where one finds the value for the integration constant C on the basis that T = pa when $\phi = \pi/2$, where p is the vertical principal stress and a is the horizontal radius of the closed curve. On the basis of symmetry, each branch of the curve supports the half of the pressure, therefore one gets:

$$T = \frac{pa \sqrt{2}}{\sqrt{[K + (K - 2) \cos 2\phi]^3}} \quad .$$

Substituting this value in the second equation of group (83), one gets

$$ds = \frac{2a\ (K - 1)\ \sqrt{2}\ d\phi}{\sqrt{[K + (K - 2)\ \cos 2\phi]^3}}$$

from which, after taking

$$ds = \rho d\phi,$$

the following equation is obtained:

$$\rho = \frac{2(K - 1)\ a\ \sqrt{2}}{\sqrt{[K + (K - 2)\ \cos 2\phi]^3}}$$

which represents radius of curvature of the ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{(K - 1)\ a^2} = 1$$

in which the two axes have the quotient

$$\frac{b}{a} = \sqrt{K - 1} \quad .$$

This was to be expected since both resulting diameters give an ellipse of the following equation:

$$x^2\sigma_y + y^2\sigma_x = \sigma_x\sigma_y \ .$$

These observations show that an elliptical arc is quite capable of withstanding the stress which is expressed by formula (21). But there is still the question as to whether this ellipse forms the limit of the flow zone, since the number of unknowns is larger than the number of equations as can be seen from the following discussion.

By taking the following function for F,

$$F = \frac{pc^2}{4} \{2\xi \cosh 2\xi_0 + e^{-2\xi} + \cos 2\eta\} , \qquad (84)$$

and using formulae (36), (37), (38), (39) and (40), one gets the stress for an elliptical opening which is under uniform inner pressure p as follows:

$$\left.\begin{aligned}
\sigma_\xi &= p - p\,\frac{\cosh 2\xi - \cosh 2\xi_o}{(\cosh 2\xi - \cos 2\eta)^2}\ \sinh 2\xi \\
\sigma_\eta &= p - p\,\frac{\sinh 2\xi}{\cosh 2\xi - \cos 2\eta}\left(1 + \frac{\cosh 2\xi_o - \cos 2\eta}{\cosh 2\xi - \cos 2\eta}\right) \\
\tau &= p\,\frac{\cosh 2\xi - \cosh 2\xi_o}{(\cosh 2\xi - \cos 2\eta)^2}\ \sin 2\eta \ .
\end{aligned}\right\} \quad (84a)$$

By substituting for $\xi = \xi_o$, then the equation of the following form:

$$\left.\begin{aligned}
\sigma_{\xi_{\xi = \xi_o}} &= p \\
\sigma_{\eta_{\xi = \xi_o}} &= p - 2p\,\frac{\sinh 2\xi_o}{\cosh 2\xi_o - \cos 2\eta} \\
\tau_{\xi = \xi_o} &= 0
\end{aligned}\right\} \quad (84b)$$

and for $\xi \to \infty$

$$\sigma_{\xi_{\xi \to \infty}} = 0 \ ; \quad \sigma_{\eta_{\xi \to \infty}} = 0 \ ; \quad \tau_{\xi \to \infty} = 0.$$

By letting P_i be the inner uniform pressure and p the larger principal stress acting at infinity, by superimposing one gets the following equation by using equation (50) and the second equation of group (84b).

$$\sigma_{\eta_{\xi = \xi_o}} = \frac{p}{m - 1}\ \frac{m \sinh 2\xi_o + (m - 2)(1 - e^{2\xi_o} \cos 2\eta)}{\cosh 2\xi_o - \cos 2\eta}$$

$$- p_i - \frac{2 \sinh 2\xi_o}{\cosh 2\xi_o - \cos 2\eta}\ p_i \ . \quad (84c)$$

Because ξ_o and p_i are unknown and one has no additional equation to use, it is impossible to solve the problem. Thus, an arbitrary number can be taken for ξ_o in the following manner. Introducing the following:

$$\sum_0^\infty A_n\,(r) \cos n\phi \ ; \quad (n = 1, 2, 3\ldots)$$

into group (80) and considering equation (81), it is possible to find a solution by using the series. Remembering that equation (81) is not linear we come to the conclusion that solving by this method would be much too tedious.

The determination of the limit of the flow zone is of minor importance in this investigation since the volume only can be determined by it, while one is to prove that an equilibrium condition is valid also for the lowest coefficient of friction. Thus it is not worth while to proceed by the indicated method and the investigation can be reasoned on a more simple basis, which consists of determining the limits of the flow zone on the symmetry axis only, and with that the limiting values.

Since both stresses have the limiting value on the symmetry axis, for $\phi = 0$, the first derivatives

$$\frac{\partial \tau}{\partial \phi} = \frac{\partial \sigma_t}{\partial \phi} = 0$$

must disappear, so that equation (79) will be simply expressed by equation:

$$\sigma_R + r \frac{\partial \sigma_R}{\partial r} - \sigma_T + r\gamma = 0 \quad . \tag{85}$$

(The same equation could also be obtained from the equilibrium condition of an element when $\phi = 0$.)

By introducing a stress function F one gets the following values for the stresses, as could be proved by differentiation.

$$\left.\begin{aligned} \sigma_R &= \frac{1}{r}\frac{\partial F}{\partial r} - \frac{\gamma r}{2} \\ \sigma_T &= \frac{\partial^2 F}{\partial r^2} \\ \tau &= 0 \end{aligned}\right\} . \tag{85a}$$

Condition (81) finally allows one to set up the following differential equation:

$$\frac{\partial^2 F}{\partial r^2} - \frac{K-1}{r}\frac{\partial F}{\partial r} + \frac{K-1}{2}\gamma r = 0 \quad .$$

After double integration one obtains the equation

$$F = \frac{C}{K} r^K + \frac{K - 1}{6(K - 3)} \cdot \gamma r^3 ,$$

but, if, for r = a one puts $\sigma_R = \sigma_{Ra}$ the stresses are of the following form:

$$\left.\begin{aligned} \sigma_R &= \sigma_{Ra} \left(\frac{r}{a}\right)^{K-2} - \frac{\gamma a}{K - 3} \left(\frac{r}{a}\right)^{K-2} + \frac{\gamma r}{K - 3} \\ \sigma_T &= (K - 1) \sigma_{Ra} \left(\frac{r}{a}\right)^{K-2} - (K - 1) \frac{\gamma a}{K - 3}\left(\frac{r}{a}\right)^{K-2} + \\ &\quad + (K - 1) \frac{\gamma r}{K - 3} \quad . \end{aligned}\right\} \quad (85b)$$

One can get the same solution without introducing the stress function if one puts $\sigma_r = (K - 1)\sigma_R$ and determines σ_R by integration. For the lowest value of the coefficient of friction $\mu = 0.35354$ one gets (see equation (72)):

$$\sigma_R = \frac{r}{a} \left(\sigma_{Ra} - \gamma a \log \frac{r}{a}\right) . \qquad (85c)$$

$\phi = \pi$ changes the sign of the part of the equation containing γ, so that σ_R has values between

$$\sigma_R = \frac{r}{a} \left(\sigma_{Ra} \mp a\gamma \log \frac{r}{a}\right) \qquad (85d)$$

for any angle of ϕ, where the negative sign is valid for $\phi = 0$ and the positive sign for $\phi = \pi$.

To eliminate the influence of γ in equation (85b), one must put

$$\frac{\partial \tau}{\partial \phi} = - r\gamma \cos \phi; \ \frac{\partial \tau}{\partial r} = \gamma \sin \phi,$$

as it can be seen from equation (79), that is,

$$\tau = - r\gamma \sin\phi + r\gamma \sin\phi = 0$$

so that the influence of γ in equations (85b) vanishes, then σ_R attains the value given by equations (58), whereon it is inferred that this value of angle ϕ represents a principal direction. This result, as well as the conditions arrived at on the y-axis, prove that the function (82) must have many parameters to find a useful approximate solution.

Considering that one finds the value of the flow zone vertically above the investigated adit or tunnel by equation (85c), one comes to the conclusion that one can also find the smallest value of σ_{Ra} which is able to produce equilibrium by equation (85c).

The sum of the two principal stresses

$$\sigma_R + \sigma_T = 3\,\frac{r}{a}\left(\sigma_{Ra} - \gamma a \log\frac{r}{a}\right)$$

must be equal to the sum of the two primary principal stresses on the boundary of the flow zone, that is

$$\frac{r}{a}\left(\sigma_{Ra} - \gamma a \log\frac{r}{a}\right) = \frac{h - r}{2}\,\gamma ,$$

and then

$$\sigma_{Ra} = \frac{ha\gamma}{2r} - \frac{\gamma a}{2} + a\gamma \log\frac{r}{a} .$$

To get the highest value for the second part $r = \frac{h}{a}$, and the stress becomes the following value:

$$\sigma_{Ra} = a\gamma\left(\frac{1}{2} + \log\frac{h}{2a}\right) .$$

In other words, to get equilibrium at all, σ_{Ra} may not be smaller than the above given value.

For $a = 3$m., $\gamma = 2.5$ and $h = 1000$ m. the value of σ_{Ra} will be 4.215 kg./cm.^{2}

With that, the proof is provided that, in this case, one can obtain equilibrium also, if only the pressure on the construction at a depth of 1000 m. and $a = 3$m., $\gamma = 2.5$, is larger or equal to 4.215 kg./cm.^{2}

For the general case, the smallest value of σ_{Ra} will be equal to

$$\sigma_{Ra\ min.} = \frac{\gamma a}{K - 3} - \frac{K^2 - 5K + 7}{(K - 1)(K - 3)} \gamma h \left[\frac{2a}{(K - 2)h}\right]^{K-2} .$$

Then the flow zone becomes of the following value:

$$r = \frac{K - 2}{2} h.$$

For K = 4 we get:

$$\sigma_{Ra\ min.} = \gamma a \left(1 - \frac{a}{h}\right) .$$

In Fig. 27 the flow zones are diagrammatically shown for $\mu = 0.35354$, a = 3m. $\gamma = 2.5$, h = 1000 m. and the following values are used for σ_{Ra}: 4.215 kg./cm.^{2}, 4.5 kg./cm.^{2}, 6.0 kg./cm.^{2}, 10.0 kg./cm.^{2} and 30 kg./cm.^{2} The calculated crown heights are 1/2 h, 1/5 h, 1/15 h, 1/22 h, and 1/100 h (approximately). It can be seen that σ_{Ra} cannot be smaller than 4.215 kg./cm.^{2} as otherwise the flow zone would be an open curve because $3\sigma_{Ra}$ can never reach the value of the sum of the primary stresses. Since the flow zone for $\sigma_{Ra} = \gamma h$ coincides with a circular collar construction, i.e., it is a circle, and thus a closed curve, and for

$$a\gamma \left(\frac{1}{2} + \log \frac{h}{2a}\right) < \sigma_{Ra} < \gamma h$$

the same also represents a closed curve, while

$$\sigma_{Ra} < a\gamma \left(\frac{1}{2} + \log \frac{h}{2a}\right)$$

the flow zone must be an open curve, it is obvious that in spite of the lack of information as to the shape of the flow zone, it can be taken as an ellipse and the adit put in the lower focus. The flow zones in Fig. 27 were drawn under this assumption, and in connection with that one puts, for numerical eccentricity and parameter p, when A and B are taken as the outer limits given by formulae (85c), the following equations:

$$\varepsilon = \frac{A - B}{A + B} \qquad p = \frac{2AB}{A + B} .$$

The equation of the ellipse will then be determined by the following formula:

$$b = \frac{2\ A\ B}{A + B - (A - B)\cos\phi} \tag{86}$$

where b represents the radius from the origin.

Expressed in rectangular coordinates the equation is:

$$\frac{4\xi^2}{(A + B)^2} + \frac{\eta^2}{AB} = 1 \quad . \tag{86a}$$

It can be seen from Fig. 27 that a small change of σ_{Ra} has a large influence on the shape of the flow zone. This condition shows that to make an exact determination of the form of a flow zone is without purpose, due to the stress conditions which are independent of the flow zone limits (i.e., those near the adit). It will be determined only by the stress σ_{Ra} and also through equation (79), whose limiting values are known.

In the case of a cohesive medium, many of the already computed results are valid if, instead of the reacting force σ_{Ra} for the circular collar construction, one uses the elastic reaction of the medium according to arch construction (when it does not crumble). In this case, one must consider three zones, namely,

1. The elastic zone outside the flow zone.
2. The flow zone itself.
3. The elastic zone at the joint.

Probably one should deal with four zones in practice. The zone at the joint may be taken as the crumbling zone since it will crumble due to the irregularity of the passages in the ground under consideration, which lies in the inner half of the ellipse determined by equation (51). Further, one can count on a zone, which is originally in the plastic condition, which will, by the flow lose the stress conditions so that the shearing stresses will be less than the break shearing stress and will therefore offer a certain resistance (as long as it does not crumble). This resistance can be designated by σ_{Ra}. Outside this zone is found the actual flow zone, i.e., the zone whose stresses increase with the distance from the origin. Finally an elastic zone should exist outside the flow zone. In this zone the tangential stresses decrease with the distance.

One can assume that a ring-formed yielding construction in cohesive media must prove to be satisfactory because it permits unstressing and yet diminishes crumbling, so that the singular stress distribution which has been investigated in this chapter pertains, and thus in a short time equilibrium can come about. However, in the case of a tunnel driven through clay at a depth of a few 1000 m. below the surface, it will be quite suitable to construct a wall which can withstand 40 to 50 kg./cm² and regulate the outer pressure through the slits so that it will not exceed the predetermined limit. At depth of 3000 m. the sum of the two primary stresses will probably not be smaller than 1000 to 1200 kg./cm²

To make a thick ring construction would not be very advisable, because then the pressure conditions would be very unfavourable. By letting the pressure drop to 40 kg./cm² then at $a = 3$, the flow zone becomes of a diameter of 60 m. (approximately) if $\gamma = 2.8$, $h = 3000$, $\mu = 0.35354$. The amount of material that has to be removed can be approximated to 5m³, which is hardly one sixth of the material removed at the beginning. But on the other hand one must remember that the tunnel walls can be of a smaller thickness. This means that there will be less material removed at the beginning.

Stress Conditions in Stratified Ground

In coal mining, the term "homogeneous ground" is not valid since the different layers which were formed at different times are not necessarily of the same composition. Since different layers may have been caused by different forces, physical properties, such as size of grains, cementing medium as well as elastic and plastic properties, may be noticeably different.

Observed from the standpoint of this investigation, a coal stratum cannot have the same properties as a sandstone and sandstone will have quite different properties from clay. To proceed with the investigation it is therefore necessary to work with a limited medium so that the assumptions correspond at least approximately to the actual circumstances.

It is known that strata lie in such a way that their border planes can be considered parallel. In the following work it will be assumed that the bordering layers lie in parallel planes. This assumption enables one to assume that the thickness of the layer under consideration is constant over the entire field of observation. On the other hand, it is assumed that the properties μ, E and m are constant for this purpose provided they are within the established limits.

It must be assumed that on border zones between different strata these values take a sudden jump and can then be taken as constant again within the new range. To establish the stress conditions, one may assume that a horizontal opening has been driven in a horizontal layer of clay whose thickness is less than the height of the opening. It is further assumed that the opening is of such a construction that the clay has the opportunity to flow into it to relieve the surrounds of stress. The layers above and below the opening will be layers of sandstone with a high cohesion and of great thickness so that, with equivalent sums of the principal stresses, they may be considered rigid as compared with the clay stratum.

Since the sandstone layers extend very far compared to the size of the opening, it is in order to say that they extend into infinity in the horizontal direction so that when a layer, taken perpendicular to the horizontal axis, is considered, we may bring the problem into a two-dimensional case. This makes the problem much simpler. For symmetrical reasons, rectangular axes will be chosen, the x-axis being in the horizontal plane of the clay layer and the y-axis in the vertical symmetrical plane. Because the clay layer is of such small thickness compared to the depth its weight can be neglected, thus making all factors containing γ disappear.

The method of solving problems so far has involved the determination of stresses by the use of the stress function F. Here it becomes difficult to follow this procedure because of condition (81); this establishes a differential equation of the second order and second degree which can be solved only with difficulty.

In this case, Hartmann's method[11] of solving this problem lends itself very well. In equation (28) if the variable y is taken for z, and the variable x taken for y, ϕ represents the angle of the principal compression direction (i.e., the direction of the largest stress) with the x-axis. Since the x-axis is now the symmetry axis, if we substitute the above-mentioned formula into the equations;

$$\left.\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau}{\partial y} &= 0 \\ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau}{\partial x} &= 0 \end{aligned}\right\} \qquad (87)$$

and if we introduce the new variable Z for σ, by means of the equation:

$$\sigma = ce^{z \cdot \tan\rho}$$

one gets the following equations:

$$\frac{\partial z}{\partial x} + \sin\rho\cos 2\phi\frac{\partial z}{\partial x} - 2\cos\rho\sin 2\phi\frac{\partial\phi}{\partial x}$$

$$+ \sin\rho\sin 2\phi\frac{\partial z}{\partial y} + 2\cos\rho\cos 2\phi\frac{\partial\phi}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} - \sin\rho\cos 2\phi\frac{\partial z}{\partial y} + 2\cos\rho\sin 2\phi\frac{\partial\phi}{\partial y}$$

$$+ \sin\rho\sin 2\phi\frac{\partial z}{\partial x} + 2\cos\rho\cos 2\phi\frac{\partial\phi}{\partial x} = 0.$$

In order to have ϕ depend on one variable only in both equations, one multiplies the first equation by $\cos\phi$, the second by $\sin 2\phi$. Adding the equations one obtains the first equation, and in a similar way the second equation, in the following way:

$$\left.\begin{aligned} \frac{\partial z}{\partial x}\cos 2\phi + \frac{\partial z}{\partial y}\sin 2\phi + \sin\rho\frac{\partial z}{\partial x} + 2\cos\rho\frac{\partial\phi}{\partial y} &= 0 \\ -\frac{\partial z}{\partial x}\sin 2\phi + \frac{\partial z}{\partial y}\cos 2\phi - \sin\rho\frac{\partial z}{\partial y} + 2\cos\rho\frac{\partial\phi}{\partial x} &= 0 \end{aligned}\right\}. \qquad (88)$$

To determine the stress condition for the given case, one assumes that the two bordering planes are laid parallel to the x-axis. At some point the principal stresses must have the angles:

$$\pm\left(\frac{\pi}{4} \pm \frac{\rho}{2}\right)$$

with the same axis since the border planes represent slip planes. In other words the angle ϕ formed by the stress trajectory with the x-axis must be independent of x and thus must be a function of y.

For this case $\phi = f(y)$, one gets the following from equation (88):

$$\left.\begin{aligned} Cy &= 2\phi\sin\rho - \sin 2\phi + C_2 \\ z\cos\rho &= \cos 2\phi + Cx + C_1 \end{aligned}\right\}, \qquad (89)$$

and the stresses attain the following values:

$$\begin{aligned}
\sigma_x &= ce^{\frac{\tan\rho}{\cos\rho}(\cos 2\phi + Cx + C_1)}(1 + \sin\rho\cos 2\phi) - c_1 \\
\sigma_y &= ce^{\frac{\tan\rho}{\cos\rho}(\cos 2\phi + Cx + C_1)}(1 - \sin\rho\cos 2\phi) - c_1 \qquad (90) \\
\tau &= ce^{\frac{\tan\rho}{\cos\rho}(\cos 2\phi + Cx + C_1)}\sin\rho\sin 2\phi
\end{aligned}$$

To determine the values of the constants, one takes the stress at the joint; cohesionless medium as σ_a. If the medium possesses cohesion then σ_a will retain the same value as long as the resistance of the collar has a smaller value than the cohesion. This is because the boundary between the elastic and the plastic media will then be determined by the value of the cohesion.

The resistance at the joint cannot be a constant but must be

$$\text{Resistance} = \sigma_a(1 + \sin\rho\cos 2\phi)e^{\frac{\tan\rho}{\cos\rho}\cos 2\phi}$$

therefore, $C_1 = -Ca$ and one gets:

$$\sigma_x = \sigma_a(1 + \sin\rho\cos 2\phi)e^{\frac{\tan\rho}{\cos\rho}[\cos 2\phi + C(x-a)]}.$$

In the determination of the constant C it should be remembered that the thickness of the layer does not come into consideration in equation (89). This condition lets one suspect that C is a function of h. Because the first of the equations (89) represents one of the equations of the stress trajectory, one can conclude that the constant C can be determined by the stress trajectory. As was already stated, ϕ represents the direction of the principal compression (pressure), therefore the following relations must be valid.

$$\frac{dy}{d\phi} : \frac{dx}{d\phi} = \tan\phi$$

and also

$$\frac{dx}{d\phi} = \frac{2(\sin\rho - \cos 2\phi)}{\tan\phi} \cdot \frac{1}{C}.$$

By integration one gets:

$$Cx = (\sin\rho - 1)\log\sin^2\phi - 2\cos^2\phi + \text{Const.}$$

One of the stress trajectories in parametric representation is:

$$\left.\begin{aligned} Cy &= 2\phi \sin\rho - \sin 2\phi + C_2 \\ Cx &= 2(\sin\rho - 1)\log\sin\phi - \cos 2\phi + \text{Const.} \end{aligned}\right\} \quad (91)$$

Because the x-axis is the symmetry axis, the larger or the smaller principal stress must coincide with it, for $y = 0, \phi = 0$ or $\phi = \frac{\pi}{2}$, which is possible only if

$$C_2' = 0 \quad \text{or} \quad C_2'' = -\pi\sin\rho.$$

Thus, it is necessary to take the two equations into consideration at the same time.

$$\left.\begin{aligned} C'y &= 2\phi\sin\rho - \sin 2\phi \\ C''y &= (2\phi - \pi)\sin\rho - \sin 2\phi \end{aligned}\right\}. \quad (91a)$$

To determine the value of C, consider that for $y = -h$ or $y = +h$, i.e. on the wall, the principal stress forms the angle

$$\left(\frac{\pi}{4} - \frac{\rho}{2}\right) \quad \text{or also} \quad -\left(\frac{\pi}{4} - \frac{\rho}{2}\right)$$

with the x-axis. By letting the first angle $(\pi/4 - \rho/2)$ increase continuously, then the value $\pi/2$ must be reached after turning through $(\pi/4 + \rho/2)$. With that it is proved that this value is valid for $C_2'' = -\pi\sin\rho$, and one gets

$$C'' = \frac{(\pi + 2\rho)\sin\rho + 2\cos\rho}{2h}.$$

Now by increasing the second angle $-(\pi/4 - \rho/2)$, continuously, one gets a value of zero after turning through $(\pi/4 - \rho/2)$. By putting this value in the first equation one gets:

$$C' = \frac{(2\rho - \pi)\sin\rho + 2\cos\rho}{2h}.$$

Thus, the value of all the constants has been determined.

To determine the second stress trajectory, one must take the relation:

$$\frac{dy}{d\phi} : \frac{dx}{d\phi} = - \cot \phi$$

or

$$C \frac{dx}{d\phi} = 2 \tan \phi \, (\sin \rho - \cos 2\phi)$$

and after integration one gets

$$Cx = 2(1 + \sin \rho) \log \cos \phi - \cos 2\phi + \text{Const.}$$

The slip lines can be determined by taking

$$\frac{dy}{d\phi} : \frac{dx}{d\phi} = \tan \left[\phi \pm \left(\frac{\pi}{4} - \frac{\rho}{2} \right) \right] \quad .$$

But because

$$\tan \left[\frac{\pi}{4} + \phi - \frac{\rho}{2} \right] = \frac{1 + \sin (2\phi - \rho)}{\cos (2\phi - \rho)}$$

one gets

$$C \frac{dx}{d\phi} = \frac{2(\sin \rho - \cos 2\phi) \cos (2\phi - \rho)}{1 + \sin (2\phi - \rho)} \; .$$

Multiplying out the numerator and dividing by the denominator one gets the equation:

$$C \frac{dx}{d\phi} = - 2 \cos \rho + 2 \sin 2\phi \; .$$

By integration one obtains:

$$Cx = -2\phi \cos \rho - \cos 2\phi + \text{Const.}$$

For the other slip line one gets:

$$Cx = 2\phi \cos \rho - \cos 2\phi + \text{Const.}$$

Summarizing, one gets the following equations for the stress trajectories and slip lines in a case where the larger principal stress makes the angle $\frac{\pi}{2}$ with the x-axis:

$$\left.\begin{aligned}
y_{1234} &= 2\,\frac{(2\phi - \pi)\sin\rho - \sin 2\phi}{(\pi + 2\rho)\sin\rho + 2\cos\rho}\,h \\
x_1 &= 2\,\frac{2(\sin\rho - 1)\log\sin\phi - \cos 2\phi}{(\pi + 2\rho)\sin\rho + 2\cos\rho}\,h + \text{Const.} \\
x_2 &= 2\,\frac{2(\sin\rho + 1)\log\cos\phi - \cos 2\phi}{(\pi + 2\rho)\sin\rho + 2\cos\rho}\,h + \text{Const.} \\
x_{34} &= 2\,\frac{\mp 2\phi\cos\rho - \cos 2\phi}{(\pi + 2\rho)\sin\rho + 2\cos\rho}\,h + \text{Const.}
\end{aligned}\right\} \quad (92)$$

and for the case where the larger principal stress coincides with the x-axis one gets:

$$\left.\begin{aligned}
y_{1234} &= 2\,\frac{2\phi\sin\rho - \sin 2\phi}{(2\rho - \pi)\sin\rho + 2\cos\rho}\,h \\
x_1 &= 2\,\frac{(\sin\rho - 1)\log\sin^2\phi - \cos 2\phi}{(2\rho - \pi)\sin\rho + 2\cos\rho}\,h + \text{Const.} \\
x_2 &= 2\,\frac{2(1 + \sin\rho)\log\cos\phi - \cos 2\phi}{(2\rho - \pi)\sin\rho + 2\cos\rho}\,h + \text{Const.} \\
x_{34} &= 2\,\frac{\mp 2\phi\cos\rho - \cos 2\phi}{(2\rho - \pi)\sin\rho + 2\cos\rho}\,h + \text{Const.}
\end{aligned}\right\} \quad (92a)\,.$$

The subscripts $_1$ and $_2$ represent the stress trajectories and the subscripts $_3$ and $_4$, the slip lines *. Equations (92) are valid for values:

$$\frac{\pi}{4} - \frac{\rho}{2} < \phi < \frac{3\pi}{4} + \frac{\rho}{2}$$

and the equations (92a) are valid for the values:

$$-\frac{\pi}{4} + \frac{\rho}{2} < \phi < \frac{\pi}{4} - \frac{\rho}{2}\,.$$

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* These formulae are after Nadai published in Handbuch der Physik, vol. VI; however they contain several typographical errors.

In Fig. 29 the stress trajectories (dotted lines) and the slip lines (solid lines) are shown for the first case. Fig. 30 shows the condition for the second case, when $\rho = 30°$.

To determine C_2 as it comes into consideration in practical mining operations, one goes through the following procedure. Because the layer is horizontal, the larger of the two principal stresses must be perpendicular to the layer, i.e., it forms an angle $\pi/2$ with the x-axis.

Considering an adit driven into a clay bank, σ_x will then drop to the smallest value at the joint itself. It is then necessary for σ_y to drop in value, and therefore there will appear a partially unstressed zone around the adit. Cracks may occur in the roof and floor within the ellipse given in equation (51).

On the other hand, from equation (53) one can see that σ_η attains the largest value for $\eta = \pi/2$ which now is determined by the value of σ_y. Thus if the clay reaches a new equilibrium condition, the larger principal stress must make an angle $\pi/2$ with the x-axis. This proves that equation (92) comes into consideration and the stresses become of the following forms:

$$\left.\begin{aligned}
\sigma_x &= \sigma_a(1 + \sin\rho\cos 2\phi) \times \\
&\quad \times \frac{\tan\rho}{e^{\cos\rho}}\left[\cos 2\phi + \frac{(2\rho+\pi)\sin\rho + 2\cos\rho}{2}\,\frac{x-a}{h}\right] \\
\sigma_y &= \sigma_a(1 - \sin\rho\cos 2\phi) \times \\
&\quad \times \frac{\tan\rho}{e^{\cos\rho}}\left[\cos 2\phi + \frac{(2\rho+\pi)\sin\rho + 2\cos\rho}{2}\,\frac{x-a}{h}\right] \\
\tau &= \sigma_a\sin\rho\sin 2\phi \times \\
&\quad \times \frac{\tan\rho}{e^{\cos\rho}}\left[\cos 2\phi + \frac{(2\rho+\pi)\sin\rho + 2\cos\rho}{2}\,\frac{x-a}{h}\right].
\end{aligned}\right\} \quad (93)$$

To get the right picture of the stress conditions, it is convenient to divide the stresses into two factors:

$$\sigma_x = \sigma_a \, (1 + \sin \rho \cos 2\phi) \, e^{\frac{\tan\rho}{\cos\rho} \cos 2\phi} \times$$

$$\times \quad e^{\frac{\tan \rho}{\cos \rho} \frac{(2\rho + \pi) \sin \rho + 2 \cos\rho}{2} \frac{x-a}{h}}$$

The first factor depends entirely on angle ϕ and therefore on y, while the second is a function of x. By putting $x = a$ the second factor will become equal to 1, and for this unfavourable case the stresses become of the following values for $K = 3$ or $\rho = 19°30'$.

$$\sigma_x = \sigma_a(1 + 0.334 \cos 2\phi) \, e^{0.376 \cos 2\phi},$$

$$\sigma_y = \sigma_a(1 - 0.334 \cos 2\phi) \, e^{0.376 \cos 2\phi},$$

$$\tau = 0.334 \, \sigma_a \sin 2\phi \, e^{0.376 \cos 2\phi} .$$

Fig. 31 shows the value of the stresses. (σ_x dotted, τ broken line, σ_y solid lines).

It can be seen at once that the largest value of each component is reached at $y = a$ or $\phi = 35°15'$. As previously mentioned, the flow zone depends on the sum of the principal stresses. The sum for $\mu = 0.35354$, ($\phi = \pi/2$) is:

$$\Sigma\sigma = 2 \, \sigma_a \times e^{0.59 \frac{x-a}{h} - 0.376} .$$

Fig. 32 shows the sum of the principal stresses diagrammatically for $h = a$.

At a depth of 1000 m. and $\gamma = 2.4$ the sum of the primary principal stresses has the value of 360 kg./cm.^{2}, so that the flow zone is (for the unfavourable case $\mu = 0.35354$ when $h = a$) extended to a distance of approximately 10.5 a in the clay bank.

For $\sigma_a = 10$ kg./cm.^{2} a value of about 5.5 a is reached, a circumstance which again gives the proof that the resistance at the joint or in other words the cohesion of the medium is of great importance. On the other hand, the thickness of the clay layer also has quite an influence on the stress condition because it appears in the exponent. In Fig. 33 the stress conditions are shown for various thicknesses of the clay layer.

It can be seen that for $h = 8\ a$, the sum of the primary principal stresses will be reached outside the diagram. This proves that by increasing the thickness of the clay layer, the stress condition approaches the condition discussed in the previous section until it can finally be represented by the laws found in that section. In Fig. 34, the values of the principal stresses are graphically represented (σ max. top, σ min. bottom) for $\rho = 19°30'$.

Summarizing, one can say that even the smallest clay layer has an influence on the stress conditions since it causes, first of all, an unstressing zone on the joints due to the low cohesion and high plasticity. Under these conditions for the same value of Poisson's number, we have a much larger crumbling zone in the roof and floor than we would have in a purely elastic ground. These conditions also point out the difficulties which would arise in computations for a thick seam of coal which contains small layers of clay. The crumbling zone would be much larger than that indicated by equation (51). Fig. 28 shows the crumbling zone.

Stress Conditions in Disturbed Ground

The stress condition in undisturbed ground is given by equation (1) which represents the largest principal stress. In disturbed ground the largest principal stress can be a function of Poisson's number and the coefficient of friction at a given point, as can be seen in Fig. 34.

Stress Conditions with a Horizontal Surface

If the ground is exposed to horizontal tectonic pressure forces (tangential forces), the primary horizontal principal stress can acquire the value p_h. In elastic ground it can be expressed by the following equation:

$$p_h = (m - 1)\ p_s$$

where p_s represents the primary vertical principal stress. This same equation is valid for a plastic medium if K is substituted for m (equation (51)). Equations (10) and (28) can also be employed unchanged in this case, if one takes p for p_h. It is understood that p_h does not have to be exactly horizontal but may be at an arbitrary angle to the direction of the tectonic force.

The pressure relations around an opening in an elastic medium would be more favourable because the stress-free zone (Fig. 35) will be on the sides rather than act vertically on the construction. In plastic media, because the flow zone has higher values, one has to deal with stresses which cause larger pressures on the same construction. In stratified ground certain strata with low coefficients of friction, such as clay or coal, which during the action of the tectonic forces have flowed, exist under high pressure so that with approach to them special precautionary measures must be adopted. If the ground is under torsion forces then the horizontal principal stresses may be equal to zero because it is known that the torsion produces tension stresses which can balance those of compression.

In the case of an opening driven normal to the principal stress, which has a value of zero, the second equation of group (40) becomes of the following form:

$$\sigma_n = \frac{\sinh 2\xi_o + 1 - e^{2\xi_o} \cos 2\eta}{\cosh 2\xi_o - \cos 2\eta}$$

that is to say the tension forces at the crown disappear only if $\xi_o = 0$, so that at the roof of the drift fractures which appear must extend theoretically to infinity, a condition, which on approaching a fault, explains observed difficulties.

In the above investigation, the surface was assumed to be horizontal, while the inclination to the horizontal at large depths can be neglected.

Stress Conditions with an Inclined Surface

In driving an opening at very shallow depths into layers of clay, dangerous stresses may be set up as can be seen from the following deliberation.

Let us take σ_x as the horizontal stress in the clay layer (Fig. 36), σ_y as the vertical stress, h as the thickness of the clay layer and y as the vertical distance between the upper surface and a point in the clay layer, which has x as the abscissa, where the origin of the coordinates is chosen on the edge of the clay, so that the vertical pressure exerted on the clay layer can be expressed by the following equation:

$$\sigma_y = \gamma y \quad .$$

Since there was a slope produced during erosion, σ_y represents the larger principal stress during the deposition of this layer. Therefore the smaller principal stress was equal to

$$\sigma_x = \frac{\gamma y}{K - 1} .$$

As soon as the clay layer was freed by erosion, the clay started to flow on the edge and therefore the stress conditions in the clay layer were basically changed.

To determine the equilibrium conditions, one must take the value σ_y from equation (93) for

$$\phi = \left(\frac{3\pi}{4} + \frac{\rho}{2}\right) .$$

This value is:

$$\sigma_y = \sigma_a \cos^2\rho \, e^{\tan^2\rho} \times$$

$$e^{\tan\rho[(2\rho + \pi)\tan\rho + 2]\frac{x-a}{2h}} .$$

Because the origin of the coordinates is on the edge of the layer, one puts x = 0, $\sigma_y = \sigma_b$, and gets:

$$\sigma_y = \sigma_b \times e^{\tan\rho[(2\rho + \pi)\tan\rho + 2]\frac{x}{2h}} \qquad (94)$$

where σ_b represents the burden of the soil bank or when one deals with a cohesive medium, it represents the sum of both resistances.

It can be seen that the stress condition will be represented by the following two equations:

$$\left.\begin{aligned} \sigma_{1y} &= \gamma y \\ \sigma_{2y} &= \sigma_b \times e^{\tan\rho[(2\rho + \pi)\tan\rho + 2]\frac{x}{2h}} \end{aligned}\right\} . \qquad (95)$$

To eliminate the double meaning, one makes the following deliberation. The first of equations (95) will be determined through the slope of the soil bank and the steep wall, while the second equation depends on the stress condition of the clay layer. The second equation is not valid as long as $\sigma_{1y} < \sigma_{2y}$, because there is no limiting condition for the stresses. But in the case of $\sigma_{1y} > \sigma_{2y}$, the clay layer cannot withstand the pressure and the clay starts to flow towards the edge. The shearing stress produced in the top layer of sandstone causes a force in the outward direction. This force may be expressed as follows:

$$\text{Force} = \tan\rho\,\sigma_b \int_0^x e^{\tan\rho[(2\rho+\pi)\tan\rho+2]\frac{x}{2h}} \times dx$$

or

$$F = \frac{2h\sigma_b}{(2\rho+\pi)\tan\rho+2}\left[e^{\tan\rho[(2\rho+\pi)\tan\rho+2]\frac{x}{2h}}\right]_0^x \qquad (96)$$

This force causes tension stresses on the top sand layers; and they can be expressed by the following formula:

$$\sigma_{tension} = \frac{2h\sigma_b}{y[(2\rho+\pi)\tan\rho+2]}\left[e^{\tan\rho[(2\rho+\pi)\tan\rho+2]\frac{x}{2h}}\right]_0^x . \qquad (97)$$

If this stress is larger than the tensile strength of the rock, it will cause a fracture in the roof, thus changing the stress conditions as given in the following deliberation. Equations (93) are valid only when sandstone banks are able to withstand the produced shearing stresses. If the roof layer is dragged along then the shearing stress disappears at the top of the clay layer and the stress condition is equal to the primary stress which exists in the lower part of the clay layer. Therefore, the new stress condition can be represented by equation (93), if one takes into consideration the values of ϕ between $\pi/4 - \rho/2$ and $\pi/2$ only and takes the double value for h. One then gets:

$$\sigma_x = \sigma_b\,(1 + \sin\rho\cos 2\phi) \times$$
$$\times\; e^{\frac{\tan\rho}{\cos\rho}\left[\cos 2\phi + \frac{(2\rho+\pi)\sin\rho + 2\cos\rho}{4}\frac{x}{h}\right]}$$

$$\left.\begin{aligned} \sigma_y &= \sigma_b\,(1 - \sin\rho\cos 2\phi)\ \times \\ &\times\ e^{\frac{\tan\rho}{\cos\rho}\left[\cos 2\phi + \frac{(2\rho+\pi)\sin\rho + 2\cos\rho}{4}\frac{x}{h}\right]} \\ \tau &= \sigma_b\,\sin\rho\sin 2\phi\ \times \\ &\times\ e^{\frac{\tan\rho}{\cos\rho}\left[\cos\rho + \frac{(2\rho+\pi)\sin\rho + 2\cos\rho}{4}\frac{x}{h}\right]} \end{aligned}\right\} \quad (98)$$

Now it is possible to determine σ_{3y}, that is, the value of the vertical stress on the upper boundary of the clay layer after the roof layer is fractured. It is

$$\sigma_{3y} = \sigma_c\, e^{\tan\rho\,[(2\rho+\pi)\tan\rho + 2]\frac{x}{4h}} \qquad (99)$$

where

$$\sigma_c = \sigma_b\,(1+\sin\rho)\, e^{-\frac{\tan\rho}{\cos\rho}} .$$

Taking now that the quotient of the two stresses at the edge

$$\frac{\sigma_b}{\sigma_c} = (1-\sin\rho)\, e^{\frac{\sin\rho}{1-\sin\rho}}$$

for clay ($\mu = 0.35354$) differs but little from unity (1.1), one comes to the conclusion that a basic change must occur in the surface in order that equilibrium may again be reached.

In Fig. 37 both equilibrium conditions are shown diagrammatically for $\sigma_b \sim \sigma_c = 1$ kg./cm.^{2}, $\gamma = 2$ and $h = 3$m. ABC represents the primary position of the soil bank and the steep wall. AD represents the primary stress condition of the clay and AE represents the limiting condition after disruption. It can be seen that the shaded part must slide away to make the tension stresses disappear in the roof. Because the vertical stress is represented in metres of earth the line AE represents the natural slope of the surface. A well in the clay layer would drop the cohesion to a fraction of a kilogram so that the stress on the edge depends only on the thickness of the talus layer.

Fig. 37 explains the huge earthslides on steep slopes caused by construction work. Construction work disrupts the roof layers and to obtain equilibrium a basic change in stress conditions must occur in the clay layer. The earthslide produces a new slope and hence new stress conditions in the surface. This can be applied to earth dams by expressing the stress conditions in the hanging earth masses by the Rankine formula, which is obtained by substituting

$$\sigma = \frac{\gamma}{\cos^2 \rho} \left[y(1 + \sin \rho \cos 2\phi) - x \sin \rho \sin 2\phi \right] \quad (100)$$

into equation (28) (ϕ is constant in this case and is therefore independent of y and x). The slope of the earth dams is a logarithmical curve whose flatness increases with decreasing load at the foot of the dam ($\sigma_b \sim \sigma_c$) and with the decrease in the coefficient of friction of the undermost layer.

Fig. 37 explains the following observed facts:

(1) the low slope angle of the topsoil with swampy underground;

(2) the destruction of an earth dam after sudden drawdown of a reservoir;

(3) invalidity of the Rankine theory for earth dams whose coefficient of friction changes;

(4) the huge slides in coal mines which are near sloping surfaces, (e.g. the Lebu Mines in Chile), and,

(5) the peculiar position of the slope after an earthslide and the low slope following equilibrium.

<u>Conclusion</u>

From the preceding computations, it can be seen that if an opening is driven in elastic ground under pressures which are below the elastic limit a singular change in stress conditions occurs. On the boundaries of the opening tension stresses occur in certain zones. These tension stresses create a stressless or slightly stressed zone within certain limits.

The elliptical stressless zone is independent of depth as well as the shape of the opening since the dimensions of this zone depend only on the width of the opening and Poisson's number. Let m be Poisson's number and b the width of the opening. The stressless zone then has the following equation:

$$\frac{4\,x^2}{(m-2)^2 b^2} + \frac{y^2}{b^2} = 1$$

The stresses at the crown of the ellipse disappear completely; but the highest value of the compression stresses can be expressed by the equation:

$$\sigma_{\eta\ max.} = \frac{m^2}{(m-2)(m-1)}\,p$$

where p represents the larger primary principal stress.

With the driving of the opening in a plastic medium, a slightly stressed zone is formed around it. The radius of this zone depends on the dimensions and the depth of the opening as well as the value of the coefficient of friction, specific weight and the pressure on the walls of the opening.

For low values of the coefficient of friction the pressure on the walls depends on the depth of the opening; but if enough material flows into the opening the pressure drops to a fraction of the primary value. For practical depths this pressure varies from 2-5 kg./cm.^{2} This shows that it is possible for the medium to reach a stress condition which is necessary to arch the opening. This condition is possible only when a certain amount of surrounding material flows into the opening.

The author can make the observation, that with earth under pressure, the pressure can be originally large enough to rupture the tunnel walls, that, however, for the same tunnel walls with slits, as soon as the arch formation in the medium is commenced and the amount of moisture of the clay which corresponds to the pressure has been reached, the earth pressure has the power to resist, a proof that only a small pressure is finally established on the tunnel walls.

In stratified ground a zone of low stresses also exists. Its dimensions depend on the thickness of the clay layer which has been cut, the coefficient of friction of the clay and on the pressure exerted on the tunnel walls.

With shallower depths and in inclined ground, basic changes in stress conditions are produced in stratified ground. This may cause large earth slides which will result in a very low natural slope.

The singular stress condition in stratified ground with inclined surface explains the difficulties observed in the construction of earth dams and permits one to draw conclusions for the construction of earth dams.

Summary

Assuming that in a homogeneous isotropic elastic medium the dilatation is proportional to the stresses and that in a homogeneous plastic medium the highest value of the shearing stress is the product of the coefficient of friction and the normal stress, we can establish a mathematical theory of ground pressures. This theory explains and proves the stressless zone in an elastic medium and the slightly stressed zone in a plastic medium when an opening is driven through the particular medium.

In stratified ground, it was shown and explained that a low stress zone exists around an opening as well as the surface inclination on the stress condition of the stratum with small coefficient of friction.

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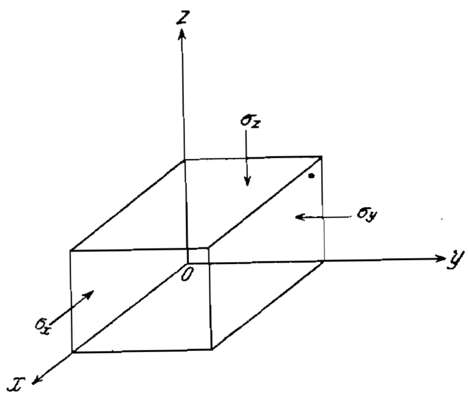

Fig. 1

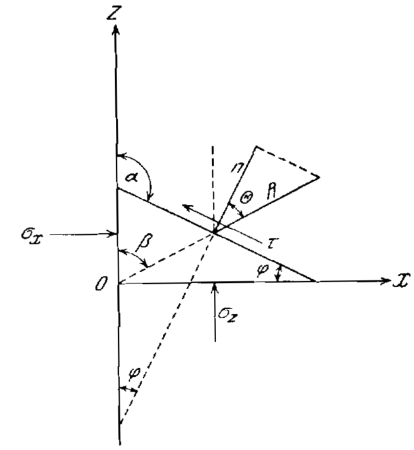

Fig. 2

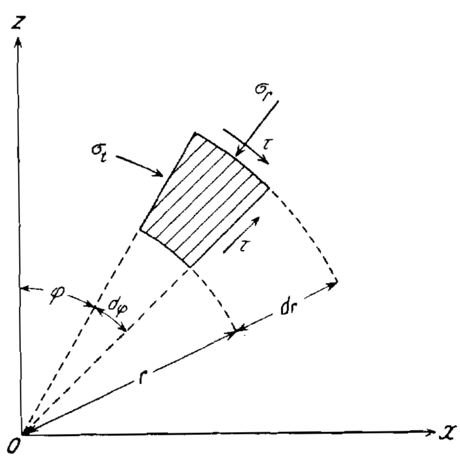

Fig. 3

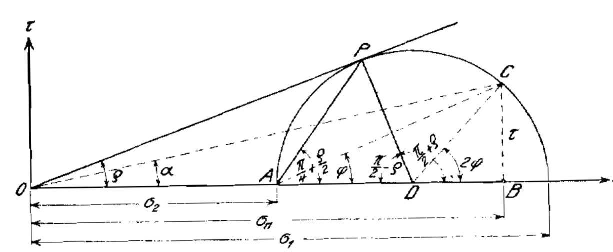

Fig. 4

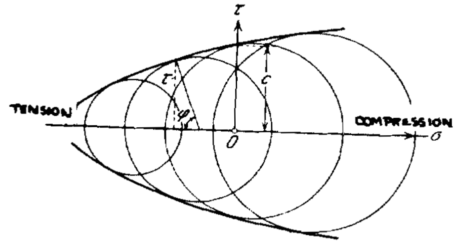

Fig. 5

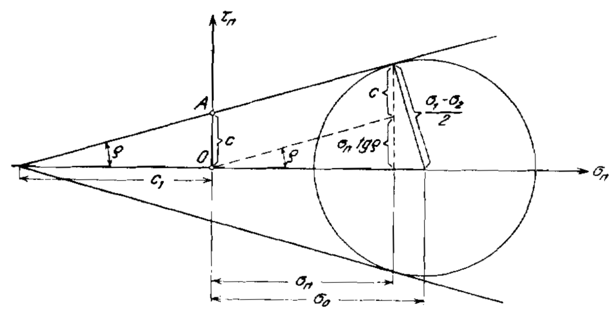

Fig. 6

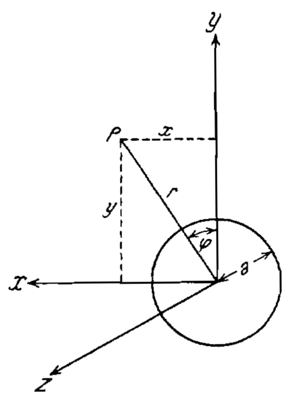

Fig. 7

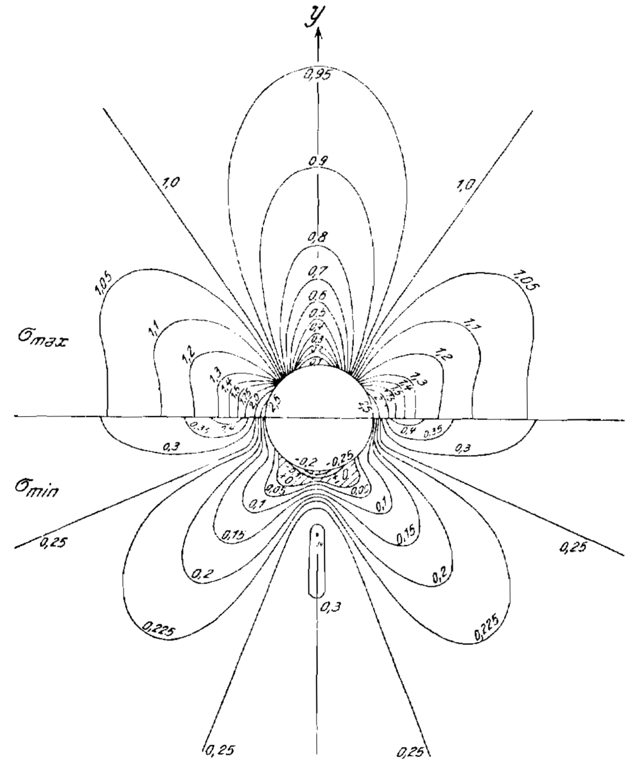

Fig. 8

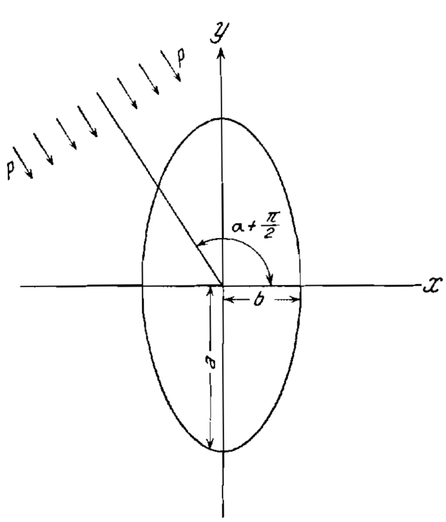

Fig. 9

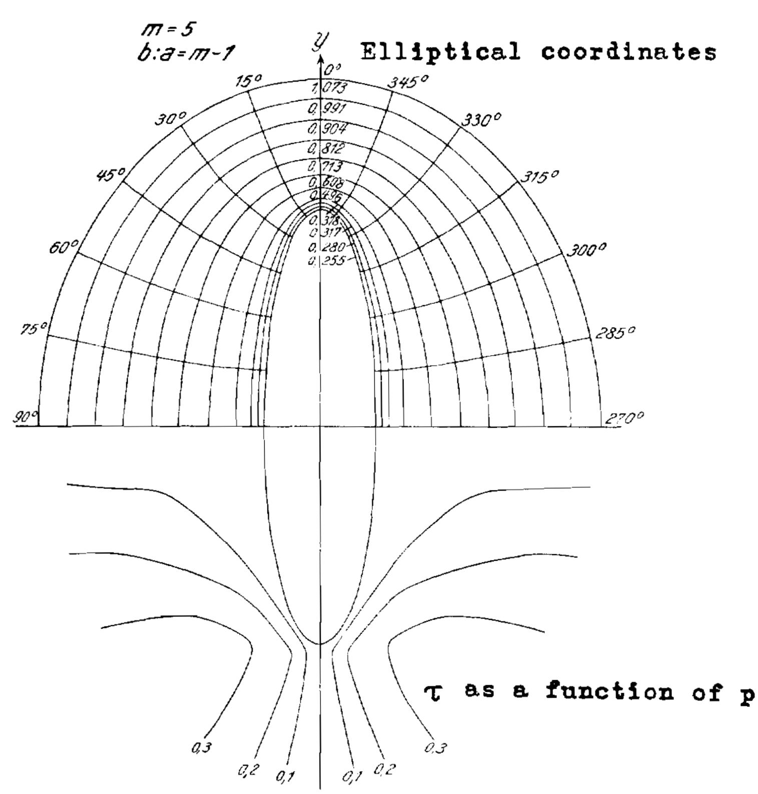

Fig. 10

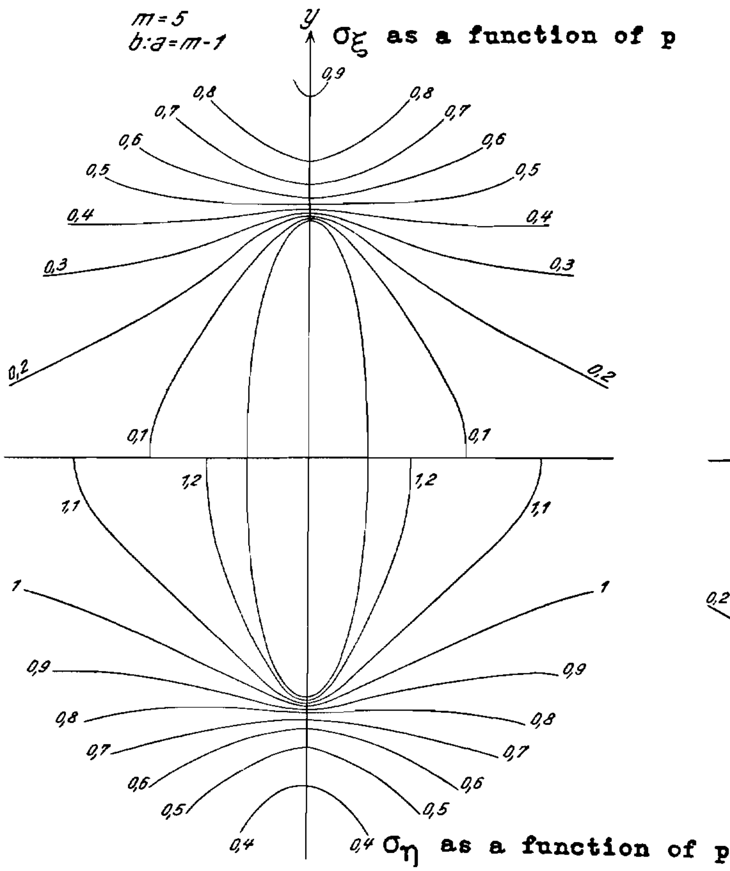

Fig. 11

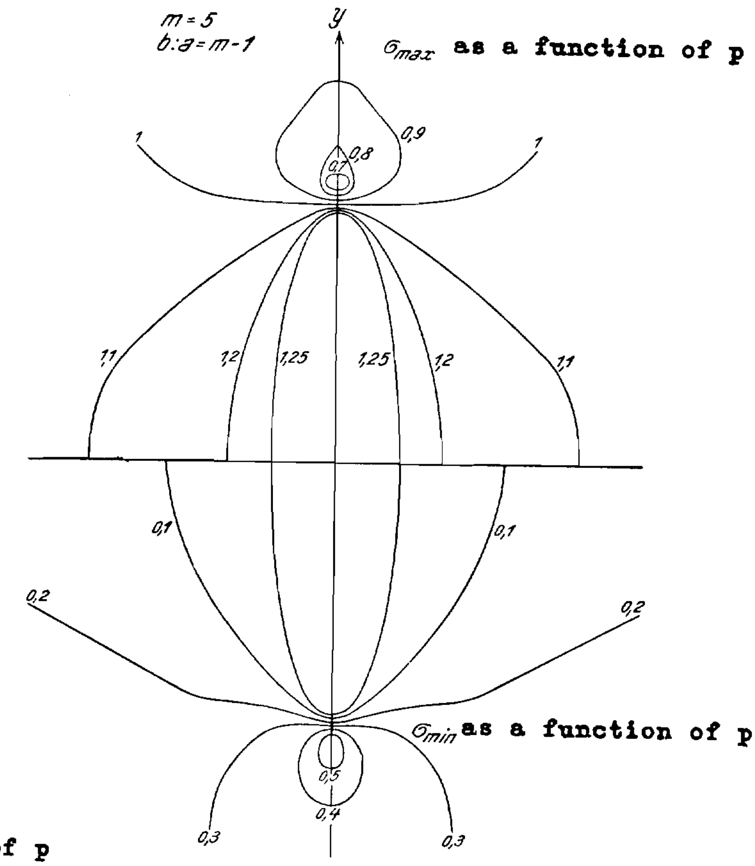

Fig. 12

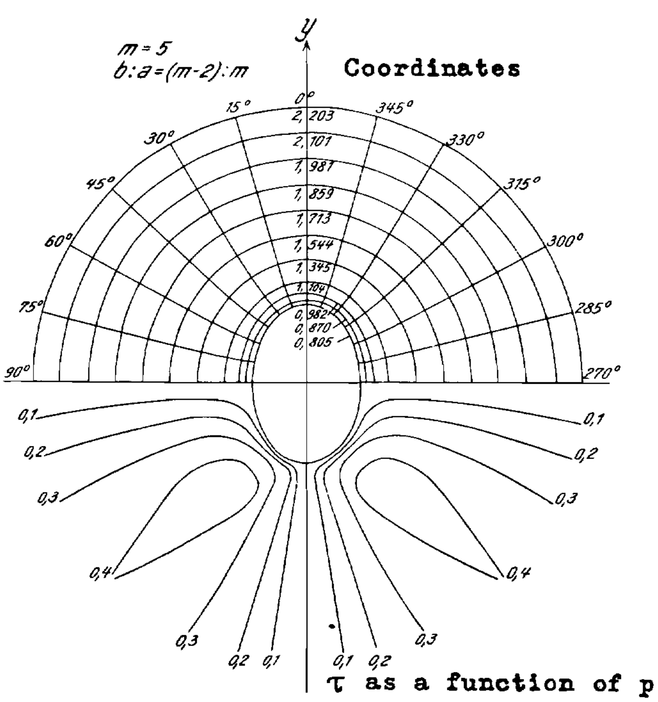

Fig. 13

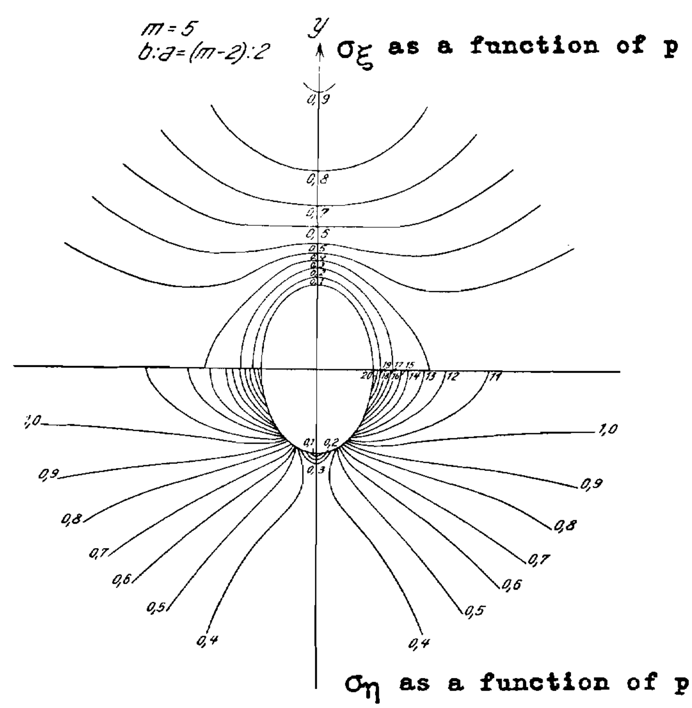

Fig. 14

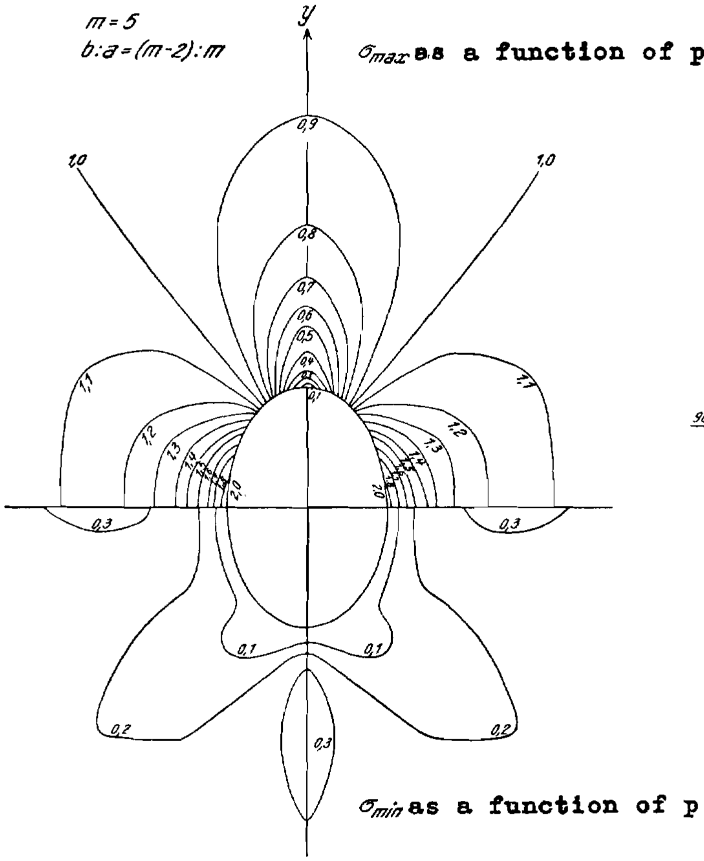

Fig. 15

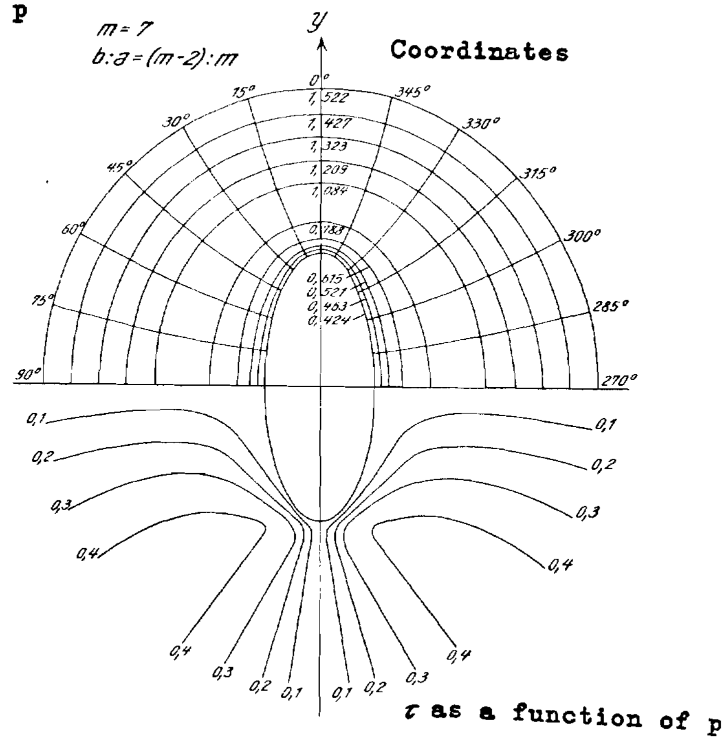

Fig. 16

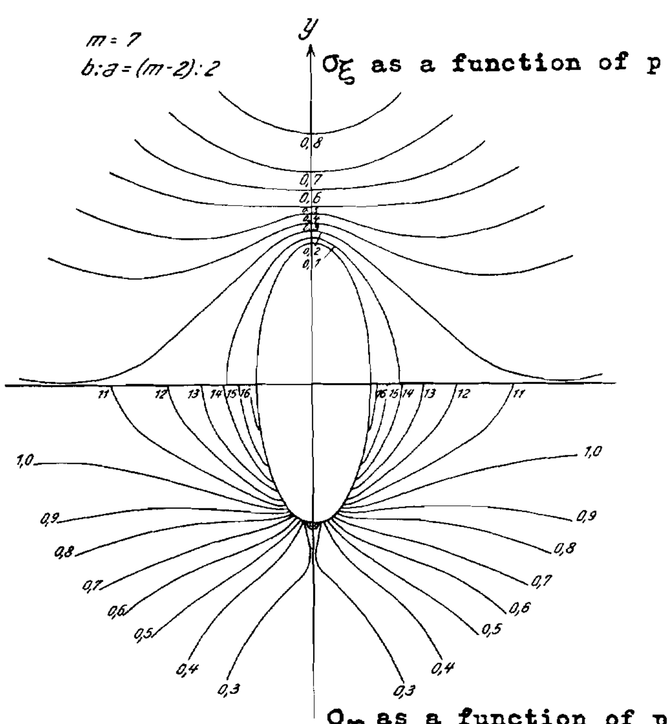

Fig. 17

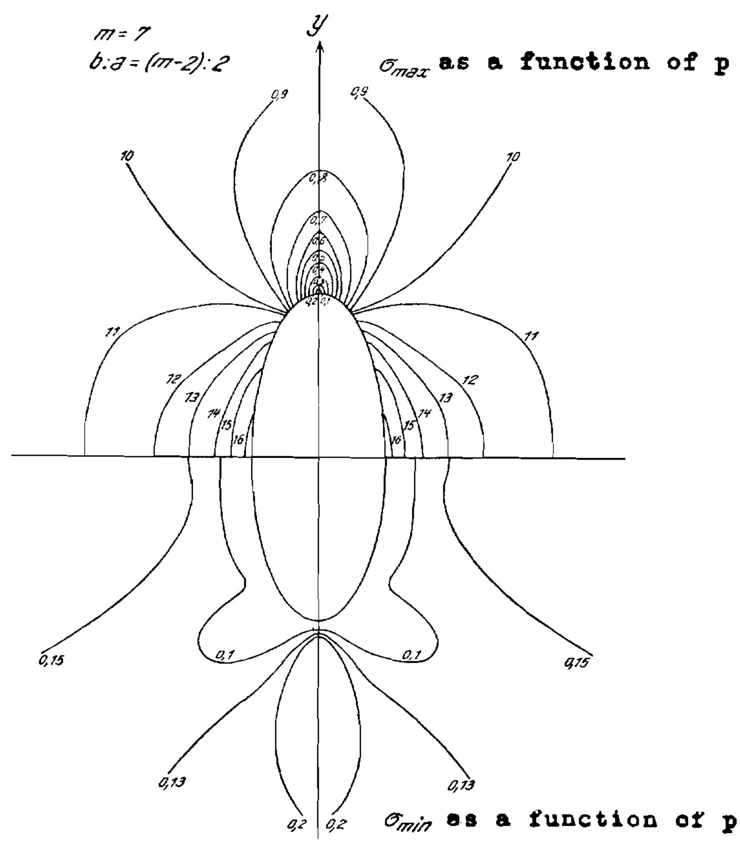

Fig. 18

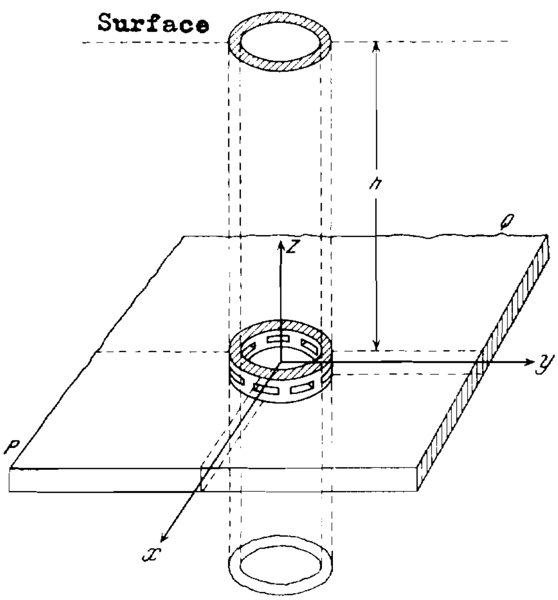

Fig. 19

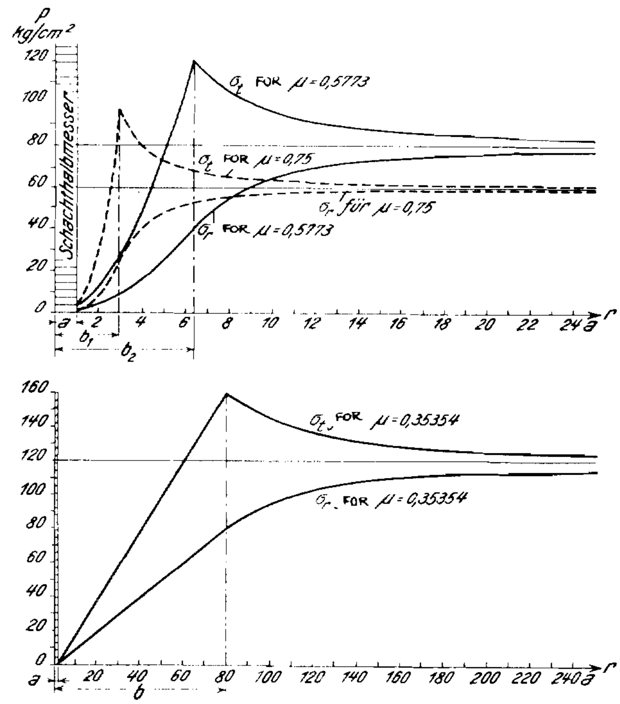

Fig. 20

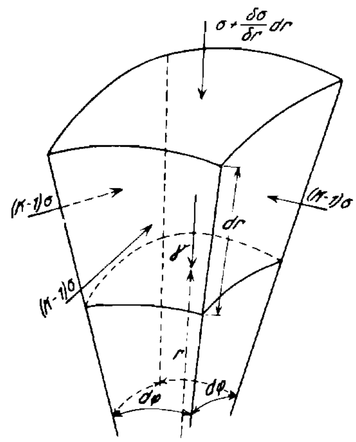

Fig. 21

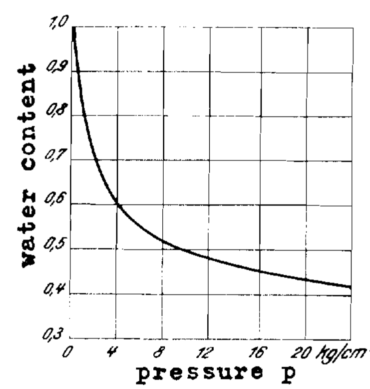

Fig. 22

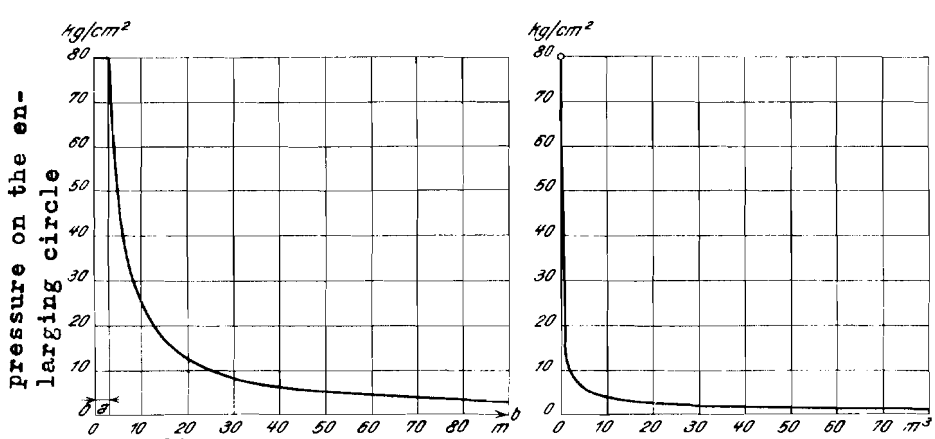

Fig. 23

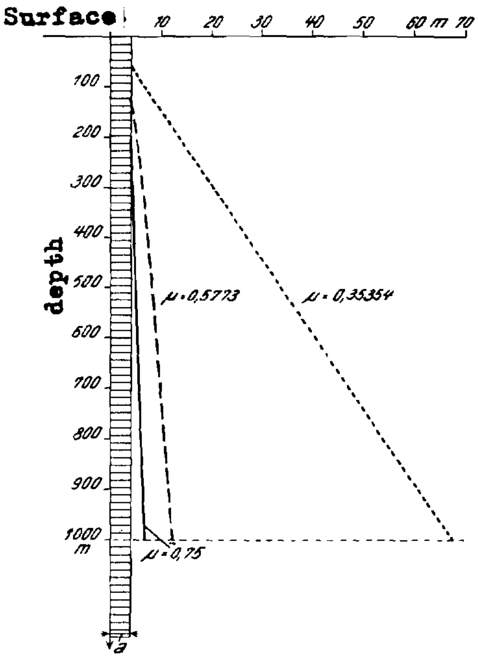

Fig. 24

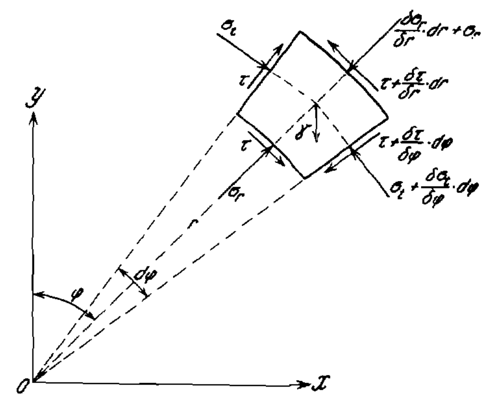

Fig. 25

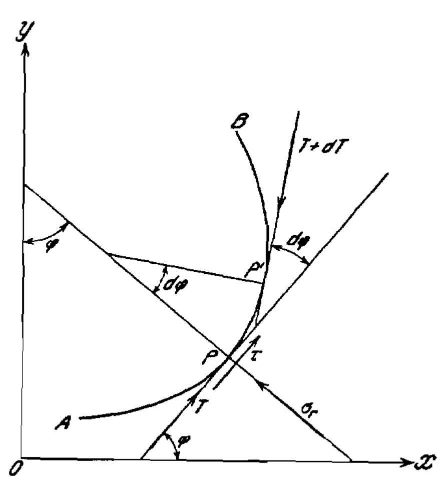

Fig. 26

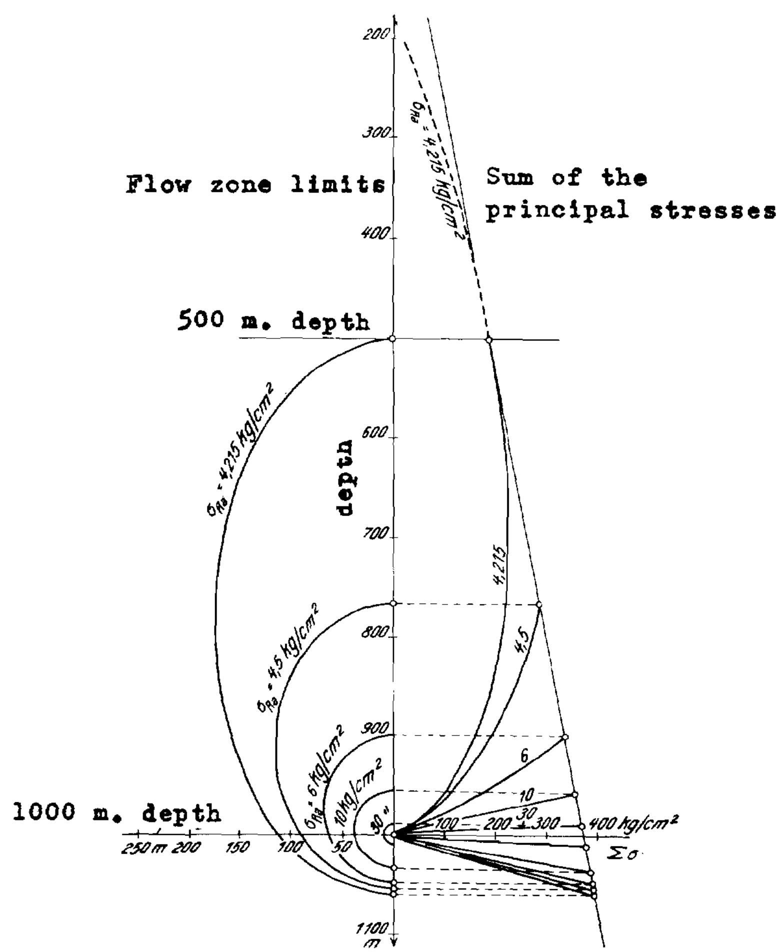

Fig. 27

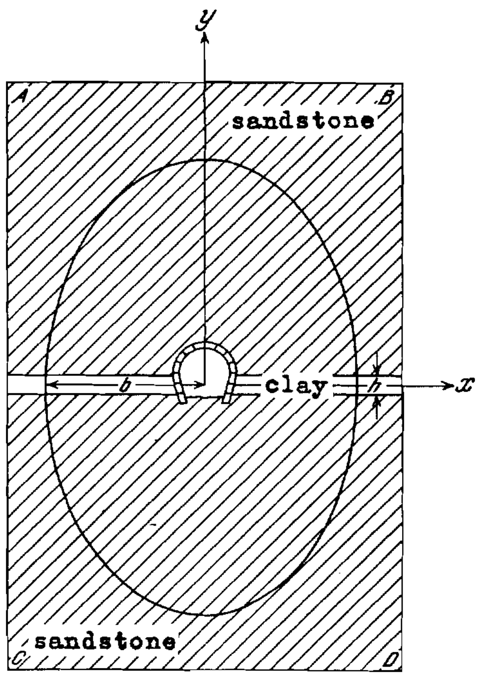

Fig. 28

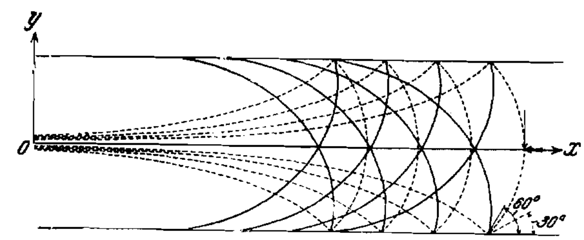

Fig. 29

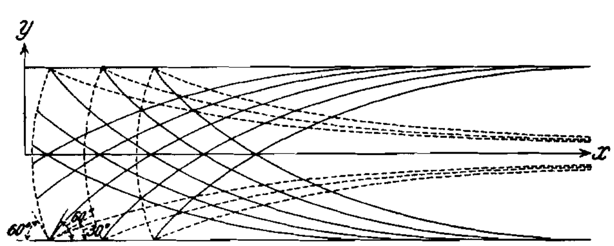

Fig. 30

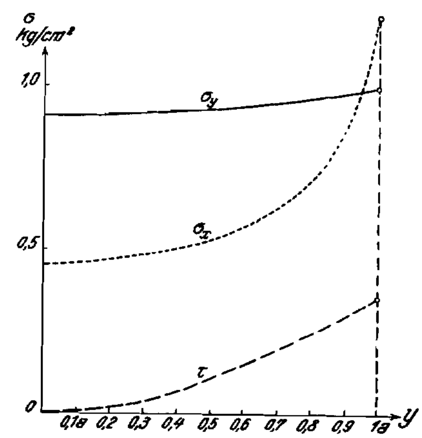

Fig. 31

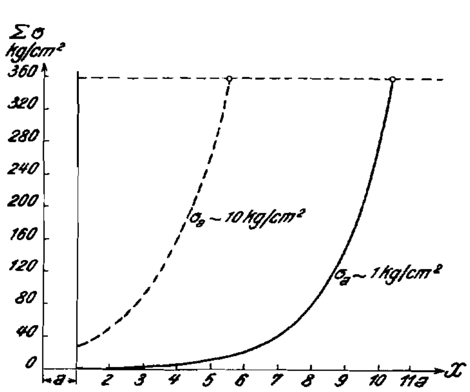

Fig. 32

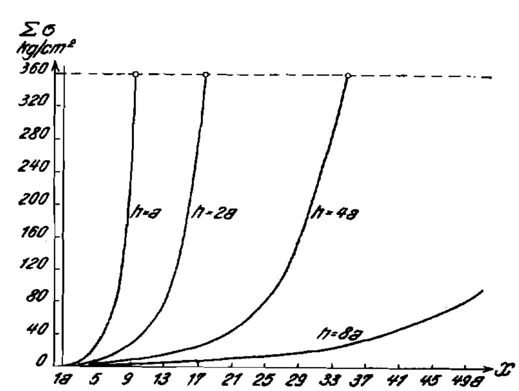

Fig. 33

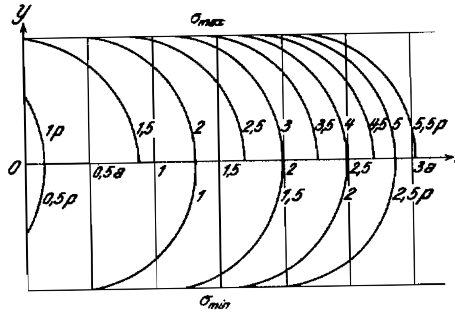

Fig. 34

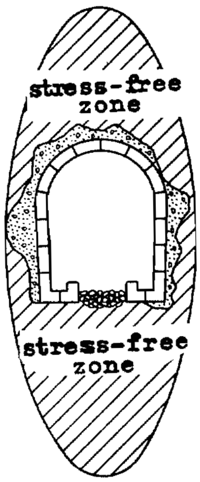

Fig. 35

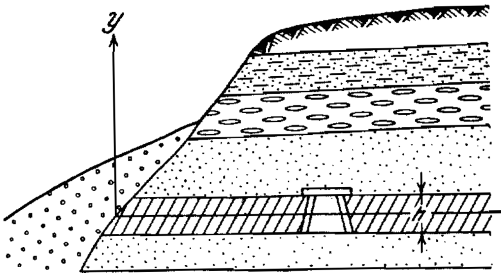

Fig. 36

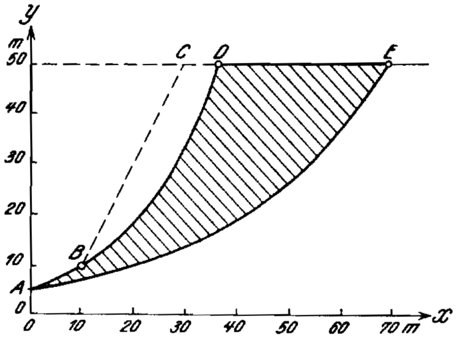

Fig. 37