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It is common practice in the construction of multi-storey buildings that have to withstand large vertical and horizontal loads to use two structural systems combined together. Usually a frame system is assumed to take most of the vertical loads and quite often a system of shear walls is designed to take the horizontal loads. In reality, however, the interaction of the two systems modifies the characteristics of the individual elements to give a composite system. The object of this paper is to present a method of structural analysis for such systems that is both simple and direct, and which could be readily applied as a design method.

The method consists of a procedure for deriving the flexibility or stiffness matrix of a structure made up of shear walls interconnected by frames. The method breaks the structure down into a number of individual parts which are either frames or shear walls. Conditions of compatibility and equilibrium are then used to set up a system of simultaneous equations, the solutions of which provide the deformations caused by a given loading condition.

A two-storey structure has been used to demonstrate the method but it can obviously be extended to any number of storeys and any configuration of shear walls and frames. Two cases of foundation support are considered namely a rigid base and a spring resistance; in the latter case differential movement can take place at ground level between the main structural components. Only rigid diaphragm action of the floors is considered here but consideration of the spring base foundation shows how this restriction can be relaxed if necessary.

Two computer programs (1, 2) are available that provide most of the information required to set up the simultaneous equations, and any structure of the type under discussion could be analysed without much delay. Eventually, however, it will be worthwhile incorporating all the individual procedures into one program that will analyse a shear wall-frame system then calculate its modes and frequencies of vibration.

METHOD OF ANALYSIS

Rigid-Base Structure

The procedure used is similar to the method of sections and can be used to study any structural system made up of different component parts. The basic idea is presented in Figure 1 for a single shear wall connected to two frame systems, when only lateral loads are considered. The procedure for dealing with any number of interconnected shear walls and frames will be shown later.

As an outcome of the applied external horizontal loads, E_i, internal forces are created in the combined system, together with deflections and rotations of the structural members. In order for the analysis to proceed, the frame components (i) and (iii) in Figure 1(b) are obtained by taking sections through the beams connected to the shear walls. When these cuts are made the internal reactions shown in Figure 1(b) must be applied to each of the component parts to maintain equilibrium.

The deflections and rotations of the cantilever shear wall can now be written down in terms of the unknown reactions P_i, V_i, M_i, as follows:

$$u_i = \sum_{j=1}^{N} \{P_j \delta_{ji} + (M_j + V_j b) \gamma_{ji}\} . \qquad \ldots (1)$$

$$\theta_i = \sum_{j=1}^{N} \{P_j \alpha_{ji} + (M_j + V_j b) \beta_{ji}\} . \qquad \ldots (2)$$

In equations (1) and (2) u_i and θ_i are the horizontal displacement and angular rotation respectively at the i^{th} floor level; δ_{ji}, γ_{ji}, α_{ji} and β_{ji} are flexibility coefficients that are easily derived, and b is the distance from the neutral axis to the extreme fibre of the shear wall; N is the number of storeys in the building.

It is also possible to write down the forces and moments that must be applied to the frame structures in order to produce given horizontal displacements of the floors and rotations of the girders connected to the shear walls. Thus if w is used to represent the lateral displacement of the floors and Ω the angular rotation of the beams which, in the combined structure, are linked to the shear walls:

$$F^{1}_{i} = \sum_{j=1}^{N} \{ w^{1}_{j} r^{1}_{ji} + \Omega^{1}_{j} S^{1}_{ji} \}$$

$$V^{1}_{i} = \sum_{j=1}^{N} \{ w^{1}_{j} t^{1}_{ji} + \Omega^{1}_{j} Z^{1}_{ji} \} \quad \{i = 1 \text{ to } N\}$$

$$M^{1}_{i} = \sum_{j=1}^{N} \{ w^{1}_{j} X^{1}_{ji} + \Omega^{1}_{j} Y^{1}_{ji} \} \qquad \ldots (3)$$

The coefficients r^{1}_{ji}, s^{1}_{ji}, t^{1}_{ji}, z^{1}_{ji}, x^{1}_{ji} and y^{1}_{ji} can readily be obtained, for example, by the moment distribution method. A similar set of equations can also be written relating F''_{i}, V''_{i} and M''_{i} to the displacements w''_{j} and Ω''_{j}.

The requirement that the structure act as a combined system introduces a set of compatibility conditions that relate the deflections and rotations of the individual components. For most practical cases it is valid to assume that the floors act as rigid diaphragms in which case each component has equal horizontal deflections when the structure is laterally loaded; it is also reasonable to assume that the rotations of the shear wall and the interconnecting beams from the frame are equal at a floor level. These restrictions on the deformations give rise to the equations:

$$u_{i} = w^{1}_{i} = w''_{i}$$

$$\text{and} \qquad \theta_{i} = \Omega^{1}_{i} = \Omega''_{i} \qquad \{i = 1 \text{ to } N\} \qquad \ldots (4)$$

If only external lateral loads are applied to the structure then application of the equilibrium conditions leads to

$$F^1_i + F''_i + P_i = E_i$$

$$V^1_i + V''_i + V_i = 0 \qquad \{i = 1 \text{ to } N\}$$

$$M^1_i + M''_i + M_i = 0. \qquad \ldots (5)$$

Equations (3), (4) and (5) can then be used to express the internal reactions acting on the cantilever in terms of the structures deformations and the stiffness coefficients of the frame components. This leads to,

$$P_i = E_i - [\sum_1^N \{u_j(r^1_{ji} + r''_{ji}) + \theta_j(s^1_{ji} + s''_{ji})\}]$$

$$V_i = - [\sum_1^N \{u_j(t^1_{ji} + t''_{ji}) + \theta_j (z^1_{ji} + z''_{ji})\}] \{i = 1 \text{ to } N\}$$

$$M_i = - [\sum_1^N \{u_j (x^1_{ji} + x''_{ji}) + \theta_j (y^1_{ji} + y''_{ji})\}] . \qquad \ldots (6)$$

Equations (6) may be substituted now into equations (1) and (2) whence a system of simultaneous equations are obtained with the u_i and θ_i as unknowns; the coefficients of the unknowns are expressed in terms of the stiffness and flexibility coefficients of the component structures, and the right hand sides of the equations have known values which are functions of the horizontal loading conditions. In order to calculate the flexibility matrix of the combined structure, a unit load is assumed at each floor level in turn, and the corresponding deflections calculated.

Example

The procedure explained above can be clarified by means of the example shown in Figure 2. In this particular case a symmetrical structure is assumed in which the members making up the frame have equal stiffness values, i.e. $I_c/h = I_{b1}/L_1 = I_{b2}/L_2$, where I is used to represent the second moment of area of a beam or column. If subscripts f and s are used to refer to the frame and shear walls respectively, and E denotes Young's Modulus, then the products E_fI_c and E_sI_s are given the symbols k_f and k_s respectively.

The flexibility coefficients in (i) and (iv) are easily obtained from the analysis of a uniform cantilever structure. There is no inherent difficulty, of course, in extending the method to include shear walls with any variation of second moment of area. It is not necessary to carry out a separate analysis of the effect of the vertical shear forces acting on the shear walls, since their action is analogous to a torque and can be described by coefficients (ii) and (iv).

In order to understand the manner in which the stiffness coefficients in (v) and (viii) are obtained it is best to follow the moment distribution approach used to obtain (v) and (vi). Thus in (v) one obtains the external forces that must be applied to displace the second floor unit distance in the horizontal direction while u_1, θ_2 and θ_1 are zero. For case (vi) θ_2 undergoes a unit rotation while u_2, u_1 and θ_1 are kept zero.

The moment distribution analyses for cases (v) and (vi) are given in Tables I and II respectively where positive moments act clockwise. In both of these tables the initial fixed-end moments caused by the imposed deflection pattern are assigned a value of 100; the relaxation process is then continued until the carry-over values are less than 1 per cent of the initial moments. The method of obtaining coefficients x_{2_j} and y_{2_j} is shown in the tables.

To calculate the other stiffness coefficients it is necessary to consider the equilibrium of the structure. For example, in case (v) the moments in columns BE and CF require equal and opposite forces acting at the 1st and 2nd floor level to maintain equilibrium. The total overturning moment of the column moments, T, is given by

$$T = M_{BE} + M_{EB} + M_{CF} + M_{FC} = -66.7 - 68.8 - 50 - 56.8 = 244.3 \qquad \ldots (7)$$

This means that there is an anticlockwise moment which must be resisted by a force r_{22} acting from left to right at the second floor where

$$r_{22}h = \frac{244.3}{100} \times \frac{6k_f}{h^2} = \frac{14.6\ k_f}{h^2}$$

or

$$r_{22} = \frac{14.6\ k_f}{h^3}$$

A similar procedure leads to the determination of the vertical shear forces that act on the cantilever when the frame is given the imposed deflections. Because of symmetry the effect of the left hand frame can be accounted for by doubling the coefficients in Figure 2.

When the information from this Figure is substituted into equations (1) through (5) the flexibility matrix of the combined structure is defined by:

$$\begin{bmatrix} (n + 26.96) & -17.10 & 11.18 & 9.25 \\ 6.94 & (n - 2.14) & 2.83 & 3.05 \\ 23.09 & -18.74 & (n + 10.92) & 6.31 \\ 14.53 & -8.13 & 5.66 & (n + 5.99) \end{bmatrix} \begin{bmatrix} \frac{u_2}{h} \\ \frac{u_1}{h} \\ \theta_2 \\ \theta_1 \end{bmatrix} = \frac{h^2}{k_f} \begin{bmatrix} 1.333 \\ 0.416 \\ 1.000 \\ 0.750 \end{bmatrix} ; \frac{h^2}{k_f} \begin{bmatrix} 0.416 \\ 0.166 \\ 0.250 \\ 0.250 \end{bmatrix}$$

Where the 1st column on the right-hand side gives the deflections in the structure caused by unit load at the second floor, and the second column (on the right) provides the deflections caused by unit load at the first floor; n is equal to $k_s/2k_f$.

The lateral deflections due to a unit load at the second floor level are shown in Figure 3 if the core wall were to stand on its own and for the combined system. The indications from these results are that even when the summation of column stiffness in the frame is only 2 per cent of the shear wall stiffness there is considerable error involved in ignoring the interaction of the frame with the core. For values less than 1 per cent the interaction can probably be ignored.

With the information obtained from the solution of equations (8) for different values of n it is possible to calculate the frequencies of vibration of the combined system. If it is further assumed that the two floor weights are equal and have the value W, then the appropriate frequencies for different values of n are shown in Table III. It can be seen in this particular case that as n becomes smaller the ratio of the two frequencies of vibration, not surprisingly, approaches the sort of value associated with a frame structure.

Extension to More Complicated Structures

It is possible to extend this approach to any distribution of shear walls and frames, including the case when the shear walls are not continuous through the height of the building. The steps involved in such an analysis are demonstrated by using the interaction system shown in Figure 4 which is loaded by the external lateral loads E_3, E_2 and E_1.

All frame members connected to a shear wall are cut and this produces three frame component systems A_1, A_2 and A_3; internal reactions F_{13}, M_{13}, V_{13} ... are placed at the cut sections, where the first subscript refers to the component number and the second subscript to the floor level. Internal reactions Q_{11}, S_{11}, R_{11} ... also act on the shear walls where the subscripts have the same meaning as for the frame internal reactions.

As a consequence of the internal reactions acting on the shear walls (which are caused by the external forces) they are deflected laterally and the cross-sections rotate. If we use u_i to represent the lateral displacement of the i^{th} floor, and θ_{ij} to represent the rotation of shear wall component i at the j^{th} floor level, it is possible to write the following sets of equations:

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} \text{xxxxxxxxx} \\ \text{xxxxxxxxx} \\ \text{xxxxxxxxx} \end{bmatrix} \begin{Bmatrix} Q_{23} \\ R_{23} \\ S_{23} \\ Q_{22} \\ R_{22} \\ S_{22} \\ Q_{21} \\ R_{21} \\ S_{21} \end{Bmatrix} = \begin{bmatrix} \text{xxxxxxxxx} \\ \text{xxxxxxxxx} \\ \text{xxxxxxxxx} \end{bmatrix} \begin{Bmatrix} Q_{33} \\ R_{33} \\ S_{33} \\ Q_{32} \\ R_{32} \\ S_{32} \\ Q_{31} \\ R_{31} \\ S_{31} \end{Bmatrix} ; \quad \ldots (9a)$$

$$u_1 = [\text{xxx}] \begin{Bmatrix} Q_{11} \\ R_{11} \\ S_{11} \end{Bmatrix} ; \quad \ldots (9b)$$

$$\theta_{11} = [\mathrm{xxx}] \begin{bmatrix} Q_{11} \\ R_{11} \\ S_{11} \end{bmatrix} ; \qquad \ldots (9c)$$

$$\begin{bmatrix} \theta_{23} \\ \theta_{22} \\ \theta_{21} \end{bmatrix} = \begin{bmatrix} \mathrm{xxxxxxxxx} \\ \mathrm{xxxxxxxxx} \\ \mathrm{xxxxxxxxx} \end{bmatrix} \begin{bmatrix} Q_{23} \\ R_{23} \\ S_{23} \\ Q_{22} \\ R_{22} \\ S_{22} \\ Q_{21} \\ R_{21} \\ S_{21} \end{bmatrix} ; \qquad \ldots (9d)$$

$$\begin{bmatrix} \theta_{33} \\ \theta_{32} \\ \theta_{31} \end{bmatrix} = \begin{bmatrix} \mathrm{xxxxxxxxx} \\ \mathrm{xxxxxxxxx} \\ \mathrm{xxxxxxxxx} \end{bmatrix} \begin{bmatrix} Q_{33} \\ R_{33} \\ S_{33} \\ Q_{32} \\ R_{32} \\ S_{32} \\ Q_{31} \\ R_{31} \\ S_{31} \end{bmatrix} . \qquad \ldots (9e)$$

The crosses shown in the matrices above are flexibility coefficients of the shear walls, which are easily obtained by such procedures as the moment-area method. The compatibility condition that all the components at any given floor level have the same displacement has been used in writing down the equations for u_i.

If we now consider the frame components, and impose the compatibility condition that at floor level i they move laterally u_i, and also that those members connected to a shear wall undergo the same rotation of the appropriate shear wall cross-section, then the values of the internal reactions of the frame can be written in terms of the unknown displacements and rotations as follows:

$$\begin{bmatrix} F_{13} \\ V_{13} \\ M_{13} \\ F_{12} \\ V_{12} \\ M_{12} \\ F_{11} \\ V_{11} \\ M_{11} \\ F^{1}_{11} \\ V^{1}_{11} \\ M^{1}_{11} \end{bmatrix} = \begin{bmatrix} \mathrm{xxxxxxx} \\ \mathrm{xxxxxxx} \\ \mathrm{xxxxxxx} \\ \mathrm{xxxxxxx} \\ \mathrm{xxxxxxx} \\ \mathrm{xxxxxxx} \\ \mathrm{xxxxxxx} \\ \mathrm{xxxxxxx} \\ \mathrm{xxxxxxx} \\ \mathrm{xxxxxxx} \\ \mathrm{xxxxxxx} \\ \mathrm{xxxxxxx} \end{bmatrix} \begin{bmatrix} u_3 \\ u_2 \\ u_1 \\ \theta_{11} \\ \theta_{23} \\ \theta_{22} \\ \theta_{21} \end{bmatrix}; \qquad \ldots (10a)$$

$$\begin{bmatrix} F_{23} \\ F_{22} \\ F_{21} \end{bmatrix} = \begin{bmatrix} \mathrm{xxxxxxxxx} \\ \mathrm{xxxxxxxxx} \\ \mathrm{xxxxxxxxx} \end{bmatrix} \begin{bmatrix} u_3 \\ u_2 \\ u_1 \\ \theta_{23} \\ \theta_{22} \\ \theta_{21} \\ \theta_{33} \\ \theta_{32} \\ \theta_{31} \end{bmatrix}; \qquad \ldots (10b)$$

$$\begin{pmatrix} V_{23} \\ V_{22} \\ V_{21} \\ V^{1}_{23} \\ V^{1}_{22} \\ V^{1}_{21} \end{pmatrix} = \begin{pmatrix} \mathrm{xxxxxxxxx} \\ \mathrm{xxxxxxxxx} \\ \mathrm{xxxxxxxxx} \\ \mathrm{xxxxxxxxx} \\ \mathrm{xxxxxxxxx} \\ \mathrm{xxxxxxxxx} \end{pmatrix} \begin{pmatrix} u_3 \\ u_2 \\ u_1 \\ \theta_{23} \\ \theta_{22} \\ \theta_{21} \\ \theta_{33} \\ \theta_{32} \\ \theta_{31} \end{pmatrix} ; \qquad \ldots (10c)$$

$$\begin{pmatrix} M_{23} \\ M_{22} \\ M_{21} \\ M^{1}_{23} \\ M^{1}_{22} \\ M^{1}_{21} \end{pmatrix} = \begin{pmatrix} \mathrm{xxxxxxxxx} \\ \mathrm{xxxxxxxxx} \\ \mathrm{xxxxxxxxx} \\ \mathrm{xxxxxxxxx} \\ \mathrm{xxxxxxxxx} \\ \mathrm{xxxxxxxxx} \end{pmatrix} \begin{pmatrix} u_3 \\ u_2 \\ u_1 \\ \theta_{23} \\ \theta_{22} \\ \theta_{21} \\ \theta_{33} \\ \theta_{32} \\ \theta_{31} \end{pmatrix} ; \qquad \ldots (10d)$$

$$\begin{pmatrix} F_{33} \\ V_{33} \\ M_{33} \\ F_{32} \\ V_{32} \\ M_{32} \\ F_{31} \\ V_{31} \\ M_{31} \end{pmatrix} = \begin{pmatrix} \mathrm{xxxxxx} \\ \mathrm{xxxxxx} \\ \mathrm{xxxxxx} \\ \mathrm{xxxxxx} \\ \mathrm{xxxxxx} \\ \mathrm{xxxxxx} \\ \mathrm{xxxxxx} \\ \mathrm{xxxxxx} \\ \mathrm{xxxxxx} \end{pmatrix} \begin{pmatrix} u_3 \\ u_2 \\ u_1 \\ \theta_{33} \\ \theta_{32} \\ \theta_{31} \end{pmatrix} ; \qquad \ldots (10e)$$

In equations (10a to 10e) the crosses represent stiffness coefficients which can be obtained from an analysis of the frame components. For example the number which multiplies the displacement u_3 in the linear expression for F_{13} is obtained by finding the force that must be applied to obtain unit displacement of the roof level of A_1 when u_2, u_1, θ_{11} θ_{23}, θ_{22} and θ_{21} are kept zero. The remaining stiffness coefficients can be interpreted in a similar manner.

To obtain an explicit evaluation of the deflections and rotations of the combined system in terms of the applied loading all that needs to be done is to obtain a sufficient number of relations between the internal reactions of the frames and the shear walls. This can be achieved by writing down the equations of equilibrium; this is perhaps best done by considering each shear wall in turn and writing down the equilibrium conditions at each floor level. Thus for S_1 the equilibrium conditions for only lateral loads become

$$\begin{aligned} Q_{11} + F^1{}_{11} - F_{11} &= E_1 \\ R_{11} + V^1{}_{11} &= 0 \\ S_{11} + M^1{}_{11} &= 0. \end{aligned} \qquad \ldots (11)$$

For S_2 at floor level 3 the equilibrium equations are

$$\begin{aligned} Q_{23} + F_{13} - F_{23} &= E_3 \\ R_{23} + V_{13} + V_{23} &= 0 \\ S_{23} + M_{13} + M_{23} &= 0. \end{aligned} \qquad \ldots (12)$$

Such a systematic procedure directly leads to a formulation of the unknown shear wall internal reactions in terms of the external loading and the unknown displacements of the combined system. The equations for the shear wall internal reactions can now be substituted into equations (9) in which case a system of simultaneous equations is obtained with the combined structure displacements as the unknowns. The coefficients of the unknowns are made up of summed products of the stiffness and flexibility coefficients of the individual components.

It is apparent from a study of equations (9) that fourteen equations can be written down, yet in fact there are only ten unknowns.

Only three equations for the lateral floor displacements are independent, however, because rigid diaphragm action has been assumed. Care must be taken therefore to ensure that, when rigid diaphragm action is assumed, independent equations are used in the final determination of the building deflections. The extra equations of course are necessary to deal with the problem of a flexible rather than a rigid connection between shear walls. The next section, although concerned with a compliant foundation restraint, provides the necessary details to handle the flexible connection case should it be required.

METHOD FOR ANALYSING THE EFFECT OF A YIELDING FOUNDATION

Unless the structure is founded on bedrock it is most likely that the component parts of a shear-frame building will be able to move relative to each other at foundation level. This situation can be dealt with quite simply if the foundation restraint can be idealized by a spring as shown in Figure 5.

In order to reduce the structural analysis involved for demonstration purposes it is assumed that the geometrical configuration and structural properties are as shown in Figure 2, except that here there is a frame on the right-hand side only of the shear wall. The foundation restraint is assumed to be represented by three springs R_1, R_2 and R_3, and this introduces two extra degrees of freedom compared with the rigid-base model. These extra degrees of freedom are represented by the motion of the foundations of the individual components u_o and w_o.

There is little change in the procedure compared with the rigid-base analysis, except that internal forces and displacements at the foundation level must be introduced, together with a means of dealing with the fact that the individual components are no longer constrained to move together at the foundation level.

For example the first step in the analysis is to determine the flexibility matrix for deflections and rotations at the levels 0, 1 and 2 of the structure shown in Figure 6a. In this particular instance it is assumed that the foundation elements are restrained in such a manner that they can only move horizontally, but it is also possible to account for rotational movements in a similar manner.

If the foundation can only move horizontally, then many of the flexibility coefficients can be obtained from Figure 2; thus the δ_{ij} coefficients are found merely by adding $1/R_1$ to those of Figure 2 whereas all the other coefficients in (i) to (iv) are unaltered. The coefficients δ_{20} , δ_{10} , δ_{02} , δ_{01} and δ_{00} are all equal to $1/R_1$ and the coefficients α_{jo}, α_{oj}, γ_{jo}, γ_{oj}, β_{jo} and β_{oj} are all zero.

The second step in the procedure is to determine the appropriate stiffness coefficients of the structural component shown in Figure 6b. Again many of these coefficients can be found from the analyses already performed for the rigid-base model. Thus r_{20} is the horizontal force acting at the foundation level of the component when u_2 has unit displacement as shown in Figure 2(v). This force can be found by summing the moments EG, GE, FH and HF then dividing by h in this way r_{20} is found to equal $3.8\ k_f/h^3$.

Because the foundation elements are assumed to be restrained so that they can only move horizontally the r_{jo} and s_{jo} coefficients are the only ones of interest for cases (v) to (viii) in Figure 2. If rotation of the foundations is to be considered then all the stiffness coefficients would have to be evaluated at the foundation level. One other condition has to be investigated and that is the stiffness coefficients mobilized when the frame foundation is given a unit horizontal displacement with the remaining degrees of freedom clamped. This last configuration to be analysed is shown in Figure 7 together with the details of the moment distribution analysis for this case. The non-zero stiffness coefficients with zero as their first subscript are shown to the right of the moment distribution table from which they are derived in Figure 7.

The equations that lead to the definition of the flexibility matrix for the structure can now be set up. Thus the equations for u_0 , u_1, u_2 , θ_1 and θ_2 are obtained from equations (1) and (2) with the exception that the summation now takes place from j equals zero to two. In exactly the same way the internal forces acting on the frame are obtained from equation (3) with the summations going from zero to two. Equation (4) remains unaltered but it is now necessary to introduce some further equations before the problem can proceed any further.

It has been assumed for the demonstration case that the foundations of the structural components cannot rotate which means that

$$\theta_0 = \Omega_0 = 0. \qquad \ldots (13)$$

We can now consider the equilibrium conditions and again equations (5) are unaltered but we need to introduce as well the equilibrium condition at the foundation level. The equations for equilibrium of vertical forces and moments need not be written down in this case because equilibrium of these reactions is automatically satisfied by the assumed foundation restraint.

If the next step, demonstrated by equation (6), is now carried out it is found that the internal forces acting on the cantilever can be found in terms of the external loading and the cantilever deflections except that w_o rather than u_o appears in the equations (the summations in the corresponding equations also go from zero to two). In order to include the effect of w_o and arrange for the coupling of the components at ground level it is necessary to consider carefully the equilibrium of horizontal forces at the foundation level. If the external force E_o is divided into two components, E_{so} and E_{fo}, where E_{so} is the force acting on the shear wall foundation and E_{fo} is the force on the frame foundation, the conditions for the equilibrium of the two foundations can be written as

$$F_o + T = E_{fo}$$

$$P_o - T = E_{so}$$

$$T = R_2 (wo-uo) \qquad \ldots (14)$$

where T is the tension in the spring connection R_2.

From equation (3) an expression for F_o can be written down:

$$F_o = \sum_{j=o}^{2} (w_j r_{jo} + \Omega_j s_{jo}) = \left[\sum_{j=1}^{2} (u_j r_{jo} + \theta_j s_{jo}) \right] + w_o r_{oo} . \qquad \ldots (15)$$

Using the last two equations of (14) with (15) it is possible to express P_o in terms of the unknown structural displacements and the external load E_{so}. By using equations (1) and (2) we can now write down five equations in terms of the six unknown displacements. The sixth equation is obtained from the first equation of (14).

It is then possible to set up the flexibility matrix of the composite structure shown in Figure 5, and the method is also readily extended to analyse a number of structural components connected by a yielding foundation. The flexibility and stiffness coefficients of the demonstration structure are shown in Tables IV and V, and the equations that result from the application of the method of analysis outlined above are shown in Table VI.

The flexibility matrix of the structure shown in Figure 5 is obtained by equating each of the external loads to unity in turn when the other remaining external loads are zero. Four such loading conditions have to be investigated, namely, $E_2 = 1$, $E_1 = 1$, $E_{so} = 1$ and $E_{fo} = 1$. Values of n equal to 10 and 50 have been investigated as they cover the range of most practical interest. The majority of results presented in this note refer to the case when $R_1/R_3 = 1.0$, $R_1/R_2 = 2.0$ and $r = 0.1$. The manner in which the flexibility coefficients vary as a function of a, which is a parameter defining the ratio of structure to foundation stiffness, is shown in Figure 8. It is interesting to note that the coefficients are linear functions of a, since this feature is not predictable beforehand. When the modes and frequencies of the structure are to be calculated the further parameters of weight distribution and damping have to be introduced. Modes and frequencies of vibration have been obtained by assuming there was no damping and that the weight of the first and second floor were equal to W, whereas the weights at the foundation level of the shear wall and frame were both W/2. Values of the fundamental frequency are given in Table VII and the ratios of the higher mode frequencies to the fundamental are plotted in Figures 9 and 10. A single example of a mode shape is shown in Figure 11.

The results for the ratio of the second to the first frequency of vibration show that there is a minimum for a value of a that appears to depend on n, but is in the vicinity of $a = 1.5$. For values of $a = 0.1$ there is a decrease in the fundamental frequency from the rigid-base curve of only 3 to 4 per cent but the second mode frequency decreases by almost 30 per cent. It is possible therefore that foundation compliance can exert an appreciable effect on modes and frequencies of vibration of a structure even if the foundation stiffness is quite high.

One set of results when $r = 0.25$ indicates that this parameter has an insignificant effect on the flexibility coefficients as did decreasing the value of R_1/R_2 from 2 to 1. Nevertheless it would be necessary to consider a larger range of values of these parameters before any general conclusions could be drawn.

Inclusion of Other Factors

Two factors that could have an appreciable effect on the dynamic characteristics of tall shear-frame structures are axial deformations and the compliance at foundation level due to base rotation. Both of these factors are easily incorporated in the method of analysis outlined in this note, and will be the subject of a future study concerned with applying this analysis procedure to a study of the dynamic characteristics of taller shear-frame structures.

CONCLUSIONS

A structural analysis procedure is described which breaks a shear-frame structure into component parts in a manner that permits well known analysis techniques such as the Moment-Area method and Moment distribution to be used in calculating the action of the composite structure. This approach gives a good physical insight into the nature of the interaction between shear walls and frames.

The results for the frequency ratios of a two-storey structure show that they are affected by the relative stiffness of both the component parts of the structure and the foundation compliances. It is now worth investigating these two basic parameters in taller shear-frame structures.

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TABLE I. MOMENT DISTRIBUTION ANALYSIS FOR CONDITIONS IN FIGURE 2(V).

Joint	AB	BA	BE	BC	CB	CF
Fixed End Moments			-100			-100
Balance		33.3	33.3	33.3	50	50
Carry Over	16.6		12.5	25	16.6	16.6
Balance		-12.5	- 12.5	-12.5	-16.6	- 16.6
Carry Over	- 6.2		- 4.1	- 8.3	- 6.2	- 6.2
Balance		4.1	4.1	4.2	6.2	6.2
Carry Over	2.0		1.5	3.1	2.0	2.1
Balance		- 1.5	- 1.5	- 1.6	- 2.0	- 2.1
$\sum$ Columns	12.4	23.4	- 66.7	43.2	50	- 50

Joint	DE	ED	EB	EF	EG	FE	FC	FH
Fixed End Moments			-100				-100	
Balance		25	25	25	25	33.3	33.3	33.3
Carry Over	12.5		16.6	16.6		12.5	25	
Balance		- 8.3	- 8.3	- 8.3	- 8.3	-12.5	-12.5	-12.5
Carry Over	- 4.1		- 6.2	- 6.2		- 4.1	- 8.3	
Balance		3.1	3.1	3.1	3.1	4.1	4.2	4.1
Carry Over	1.5		2.0	2.0		1.5	3.1	
Balance		- 1.0	- 1.0	- 1.0	- 1.0	- 1.5	- 1.6	- 1.5
$\sum$ Columns	9.9	18.8	- 68.8	31.2	18.8	33.3	-56.8	-23.4

Joint	GE	HF
Carry Over	12.5	16.6
Balance		
Carry Over	- 4.1	- 6.2
Balance		
Carry Over	1.5	2.0
Balance		
$\sum$ Columns	9.9	12.4

$$\text{A moment value of } 100 = \frac{6E_f I_c u_2}{h^2} = \frac{6k_f u_2}{h^2}$$

Thus the coefficients X_{22} and X_{21} can be obtained directly from the numbers above since they are the moments M_{AB} and M_{DE} respectively when $u_2 = 1$.

$$\text{Thus } X_{22} = \frac{12.4}{100} \times \frac{6k_f}{h^2} = \frac{0.744k_f}{h^2}$$

By considering equilibrium of the structure it is possible to calculate r_{2j} and t_{2j}.

TABLE II. MOMENT DISTRIBUTION ANALYSIS FOR CONDITIONS IN FIGURE 2(VI).

Joint	AB	BA	BE	BC	CB	CF
Fixed End Moment Balance	100					
Carry Over Balance		50 -16.6	-16.6	-16.6		
Carry Over Balance	-8.3				-8.3 4.2	4.1
Carry Over Balance		- 1.0	1.0 - 1.0	2.0 - 1.0		
$\sum$ Columns	91.7	32.4	-16.7	-15.7	-4.1	4.1

Joint	DE	ED	EB	EF	EG	FE	FC	FH
Carry Over Balance		2.1	-8.3 2.0	2.1	2.1			
Carry Over Balance	1.0					1.0 -1.0	2.0 -1.0	-1.0
$\sum$ Columns	1.0	2.1	-6.3	2.1	2.1		1.0	-1.0

Joint	GE	HF
Carry Over	1.0	-0.5
$\sum$ Columns	1.0	-0.5

$$\text{A moment value of } 100 = \frac{4E_f I_{b2} \theta_2}{L_2} = \frac{4k_f \theta_2}{h}$$

In this instance the moments M_{AB} and M_{DE} represent the coefficients y_{22} and y_{21} respectively when $\theta_2 = 1$.

$$\text{Thus } y_{22} = \frac{-8.3}{100} \times \frac{4k_f}{h} = \frac{3.67\, k_f}{h}$$

It is also possible to calculate s_{2j}, z_{2j} from the table above.

TABLE III. FREQUENCIES OF THE RIGID BASED STRUCTURE.

$\frac{ks}{2k_f} = n$	10	50	100	300
$\frac{\text{Fundamental Frequency, Hz}}{\sqrt{k_f g/Wh^3}}$	0.42	0.60	0.76	1.20
$\frac{\text{Fundamental Frequency}}{\text{Second Mode Frequency}}$	4.10	5.40	6.06	6.45

TABLE IV. FLEXIBILITY COEFFICIENTS FOR THE SHEAR WALL COMPONENTS OF FIGURE 5.

$\delta_{22} = \frac{8h^3}{3k_s} + \frac{1}{R_1}$	$\delta_{12} = \frac{5h^3}{6k_s} + \frac{1}{R_1}$	$\delta_{02} = \frac{1}{R_1}$	$\alpha_{22} = \frac{2h^2}{k_s}$	$\alpha_{12} = \frac{h}{2k_s}$	$\alpha_{02} = 0$
$\delta_{21} = \frac{5h^3}{6k_s} + \frac{1}{R_1}$	$\delta_{11} = \frac{h^3}{3k_s} + \frac{1}{R_1}$	$\delta_{01} = \frac{1}{R_1}$	$\alpha_{21} = \frac{3h^2}{2k_s}$	$\alpha_{11} = \frac{h}{2k_s}$	$\alpha_{01} = 0$
$\delta_{20} = \frac{1}{R_1}$	$\delta_{10} = \frac{1}{R_1}$	$\delta_{00} = \frac{1}{R_1}$	$\alpha_{20} = 0$	$\alpha_{10} = 0$	$\alpha_{00} = 0$
$\gamma_{22} = \frac{2h^2}{k_s}$	$\gamma_{12} = \frac{3h^2}{2k_s}$	$\gamma_{02} = 0$	$\beta_{22} = \frac{2h}{k_s}$	$\beta_{12} = \frac{h}{k_s}$	$\beta_{02} = 0$
$\gamma_{21} = \frac{h^2}{2k_s}$	$\gamma_{11} = \frac{h^2}{2k_s}$	$\gamma_{01} = 0$	$\beta_{21} = \frac{h}{k_s}$	$\beta_{11} = \frac{h}{k_s}$	$\beta_{01} = 0$
$\gamma_{20} = 0$	$\gamma_{10} = 0$	$\gamma_{00} = 0$	$\beta_{20} = 0$	$\beta_{10} = 0$	$\beta_{00} = 0$

TABLE V. STIFFNESS COEFFICIENTS FOR THE FRAME COMPONENT OF FIGURE 5.

$r_{22} = \frac{14.6\ k_f}{h^3}$	$r_{12} = \frac{-18.4\ k_f}{h^3}$	$r_{02} = \frac{3.8\ k_f}{h^3}$	$t_{22} = \frac{2.1\ k_f}{h^3}$	$t_{12} = \frac{-2.4\ k_f}{h^3}$	$t_{02} = \frac{0.2\ k_f}{h^3}$
$r_{21} = \frac{-18.4\ k_f}{h^3}$	$r_{11} = \frac{41.6\ k_f}{h^3}$	$r_{01} = \frac{-23.2\ k_f}{h^3}$	$t_{21} = \frac{0.1\ k_f}{h^3}$	$t_{11} = \frac{0.2\ k_f}{h^3}$	$t_{01} = \frac{-1.9\ k_f}{h^3}$
$r_{20} = \frac{3.8\ k_f}{h^3}$	$r_{10} = \frac{-23.2\ k_f}{h^3}$	$r_{00} = \frac{19\ k_f}{h^3} + R_3$	$t_{20} = 0$	$t_{10} = 0$	$t_{00} = 0$
$x_{22} = \frac{0.7\ k_f}{h^2}$	$x_{12} = \frac{-0.8\ k_f}{h^2}$	$x_{02} = \frac{0.07\ k_f}{h^2}$	$s_{22} = \frac{0.7\ k_f}{h^2}$	$s_{12} = \frac{0.56\ k_f}{h^2}$	$s_{01} = 0$
$x_{21} = \frac{0.6\ k_f}{h^2}$	$x_{11} = \frac{0.06\ k_f}{h^2}$	$x_{01} = \frac{-0.66\ k_f}{h^2}$	$s_{21} = \frac{-0.7\ k_f}{h^2}$	$s_{11} = \frac{0.08\ k_f}{h^2}$	$s_{01} = 0$
$x_{20} = 0$	$x_{10} = 0$	$x_{00} = 0$	$s_{20} = 0$	$s_{10} = \frac{-0.64\ k_f}{h^2}$	$s_{00} = 0$
$z_{22} = \frac{5.0\ k_f}{h^2}$	$z_{12} = \frac{0.01\ k_f}{h^2}$	$z_{02} = 0$	$y_{22} = \frac{3.7\ k_f}{h}$	$y_{12} = \frac{0.04\ k_f}{h}$	$y_{02} = 0$
$z_{21} = \frac{0.01\ k_f}{h^2}$	$z_{11} = \frac{5.2\ k_f}{h^2}$	$z_{01} = 0$	$y_{21} = \frac{0.04\ k_f}{h}$	$y_{11} = \frac{3.7\ k_f}{h}$	$y_{01} = 0$
$z_{20} = 0$	$z_{10} = 0$	$z_{00} = 0$	$y_{20} = 0$	$y_{10} = 0$	$y_{00} = 0$

TABLE VI. THE EQUATIONS DEFINING THE FLEXIBILITY COEFFICIENTS OF THE COMPOSITE STRUCTURE IN FIGURE 5.

$u_o\left(n\frac{R_2}{R_1}\right) + u_2\left(n + 25.86 + 4.35r - 3.8a\right) + u_1\left(-15.93 - 4.50r + 23.2a\right) + w_o\left(-10.0 - n\frac{R_2}{R_1} - 2.45r - 19.4a\right) + h\theta_2\left(8.74 + 10.01r\right) + h\theta_1\left(7.19 + 7.82r + 0.64a\right) = n\left[\delta_{22}E_2 + \delta_{12}E_1 + \delta_{02}E_{so}\right]$
$u_o\left(n\frac{R_2}{R_1}\right) + u_2\left(6.70 + 1.10r - 3.8a\right) + u_1\left(n - 1.91 - 1.10r + 23.2a\right) + w_o\left(-4.80 - n\frac{R_2}{R_1} - 0.85r - 19.4a\right) + h\theta_2\left(2.22 + 2.50r\right) + h\theta_1\left(2.36 + 2.60r + 0.64a\right) = n\left[\delta_{21}E_2 + \delta_{11}E_1 + \delta_{01}E_{so}\right]$
$u_o\left(1 + \frac{R_2}{R_1}\right) + u_2\left(-3.8\frac{a}{n}\right) + u_1\left(23.2\frac{a}{n}\right) + w_o\left(-19.4\frac{a}{n} - \frac{R_2}{R_1}\right) + h\theta_2(o) + h\theta_1\left(0.64\frac{a}{n}\right) = \delta_{20}E_2 + \delta_{10}E_1 + \delta_{00}E_{so}$
$u_o(o) + u_2\left(22.0 + 4.30r\right) + u_1\left(-17.54 - 4.6r\right) + w_o\left(-4.52 - 1.50r\right) + h\theta_2\left(n + 8.49 + 10.01r\right) + h\theta_1\left(4.94 + 5.22r\right) = hn\left[\alpha_{22}E_2 + \alpha_{12}E_1 + \alpha_{02}E_{so}\right]$
$u_o(o) + u_2\left(14.0 + 2.2r\right) + u_1\left(-7.54 - 2.2r\right) + w_o\left(-6.49 - 1.7r\right) + h\theta_2\left(4.44 + 5.16r\right) + h\theta_1\left(n + 4.62 + 5.21r\right) = hn\left[\alpha_{21}E_2 + \alpha_{11}E_1 + \alpha_{01}E_{so}\right]$
$-u_o\left(\frac{R_2h^3}{k_f}\right) + u_2(3.8) + u_1(-23.2) + w_o\left(19.4 + (R_2 + R_3)\frac{h^3}{k_f}\right) + h\theta_2(o) + h\theta_1(-0.64) = E_{fo}\frac{h^3}{k_f}$
where $a = \frac{k_s}{R_1h^3}$; $n = \frac{k_s}{k_f}$; $r = \frac{b}{h}$

TABLE VII FUNDAMENTAL FREQUENCIES OF THE SPRING BASED STRUCTURE OF FIGURE 5.

$n = k_s/k_f$	$a = k_s/R_1 h^3$						
	10	5	1	0.5	0.1	0.001	0
10	0.128	0.175	0.352	0.472	0.556	0.579	0.580
50	0.277	—	0.643	—	0.819	—	0.835

The 3-digit numbers in the table when multiplied by $\sqrt{\frac{k_f g}{Wh^3}}$ give the value of the fundamental frequency in Hz. for the corresponding values of a and n.

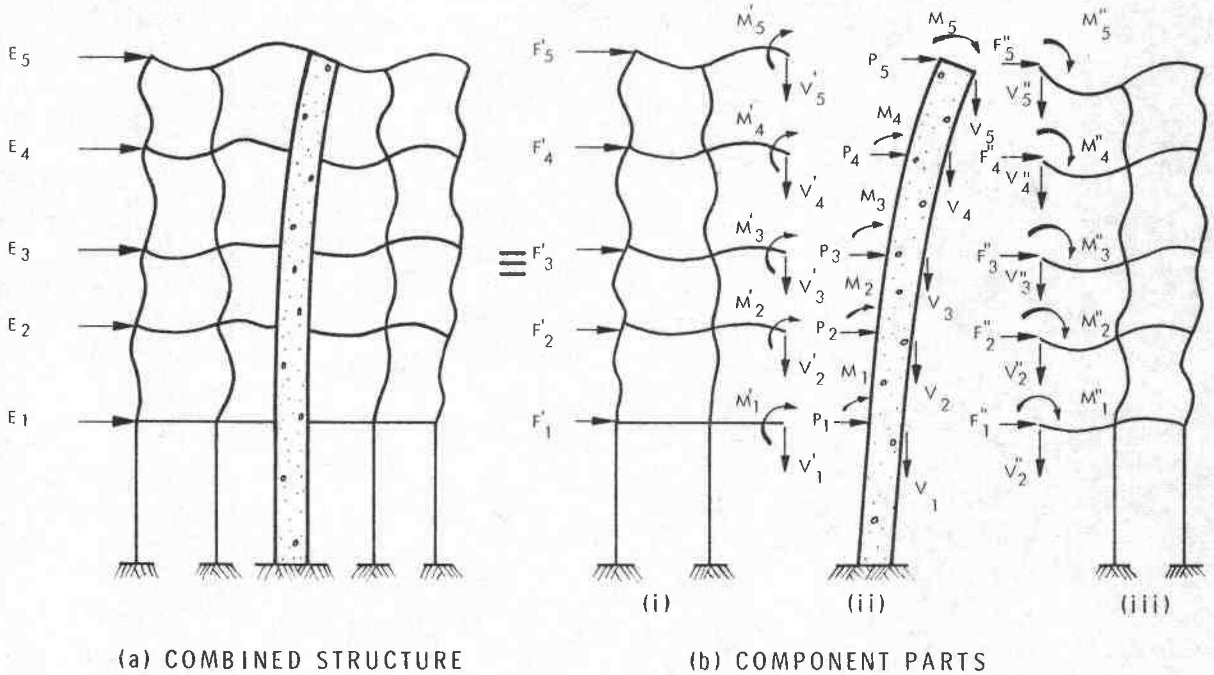

FIGURE 1 A SHEAR-FRAME STRUCTURE AND ITS COMPONENT PARTS

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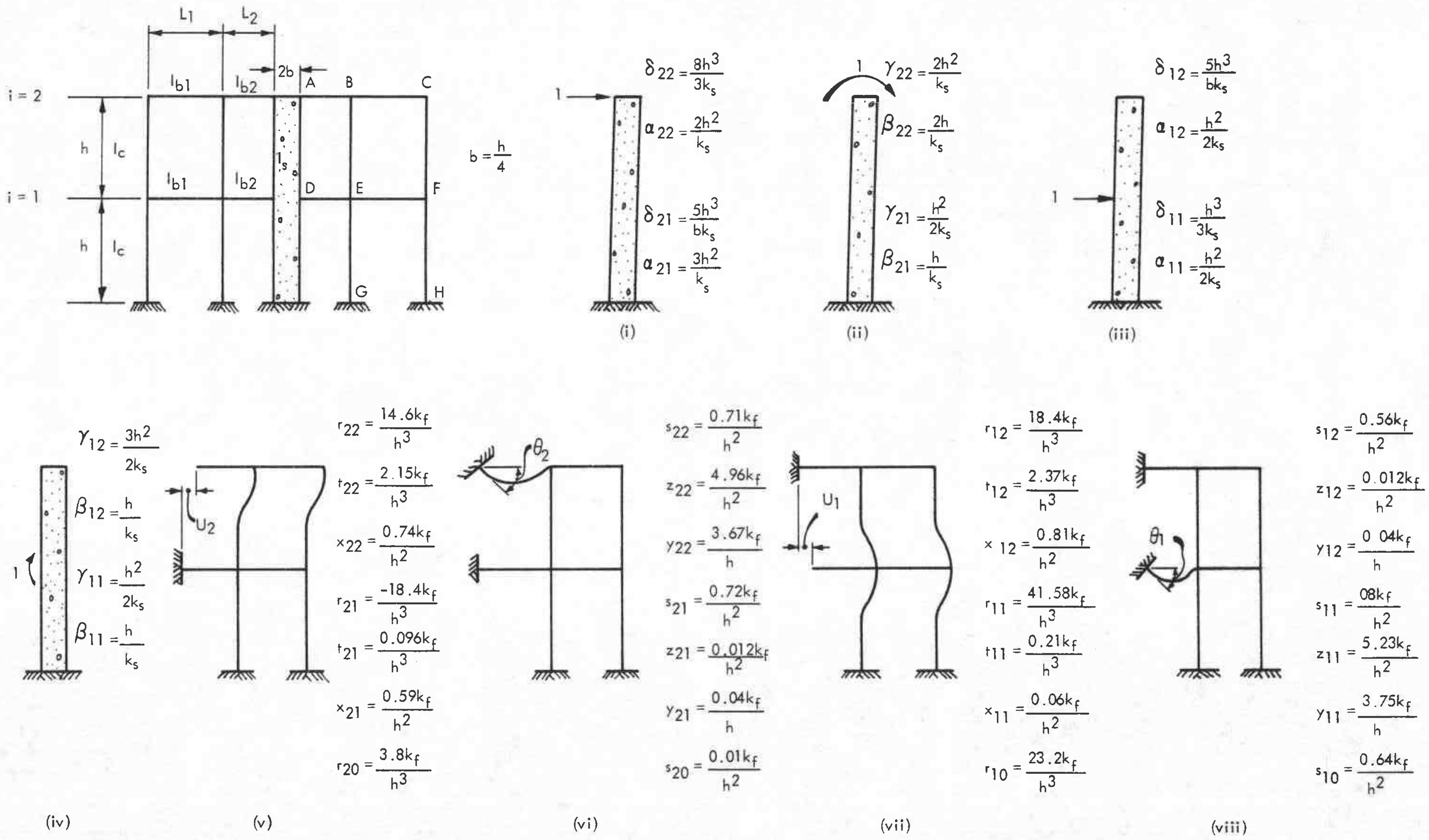

FIGURE 2

THE STIFFNESS AND FLEXIBILITY COEFFICIENTS OF THE COMPONENT PARTS OF A TWO STOREY COMPOSITE STRUCTURE

344632-2

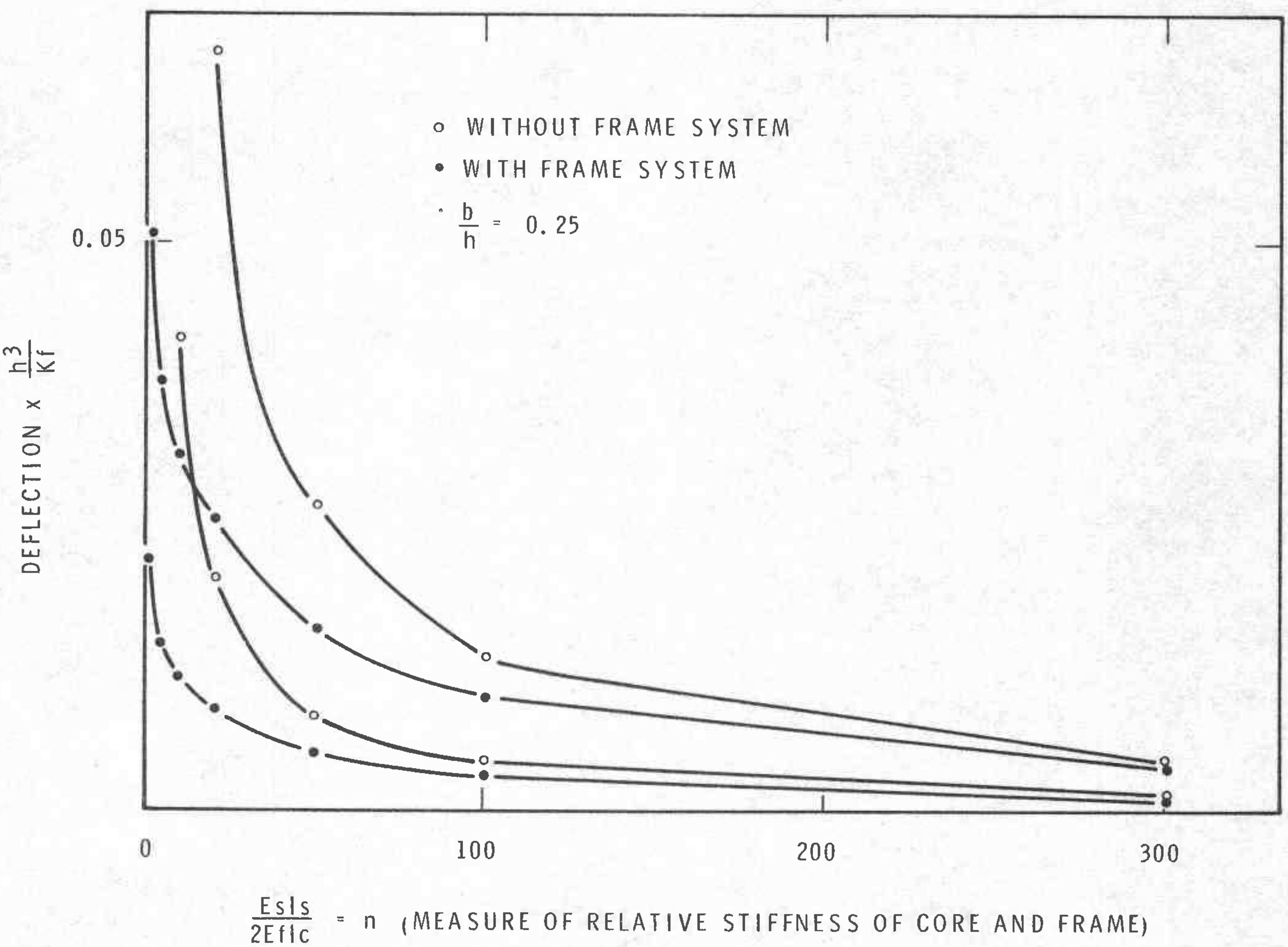

FIGURE 3

FLEXIBILITY COEFFICIENTS OF THE COMPOSITE STRUCTURE IN FIGURE 2 COMPARED WITH THOSE OF THE SHEAR WALL COMPONENT

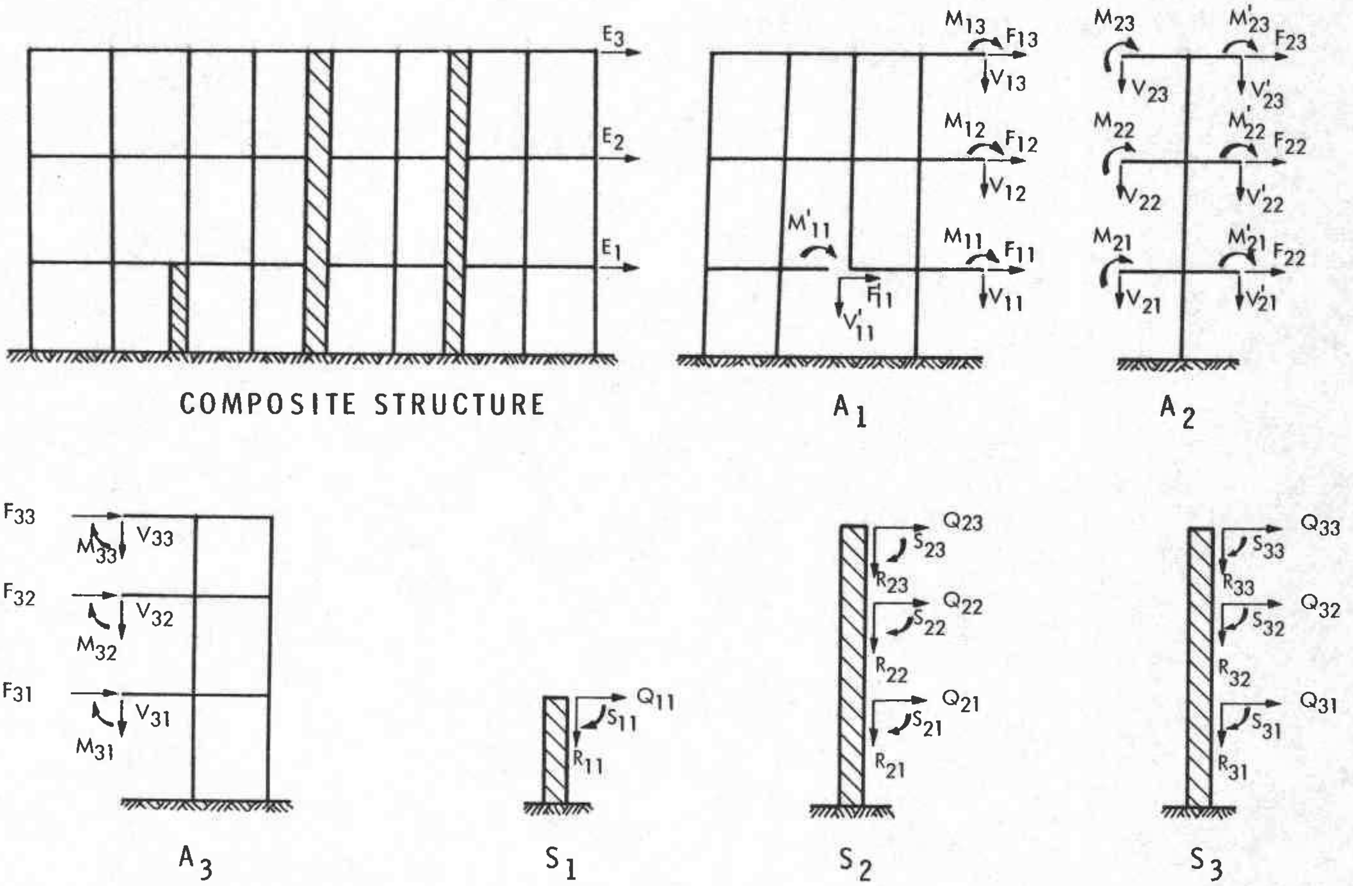

FIGURE 4

EXTENSION OF THE METHOD OF STRUCTURAL ANALYSIS TO MORE COMPLEX STRUCTURES

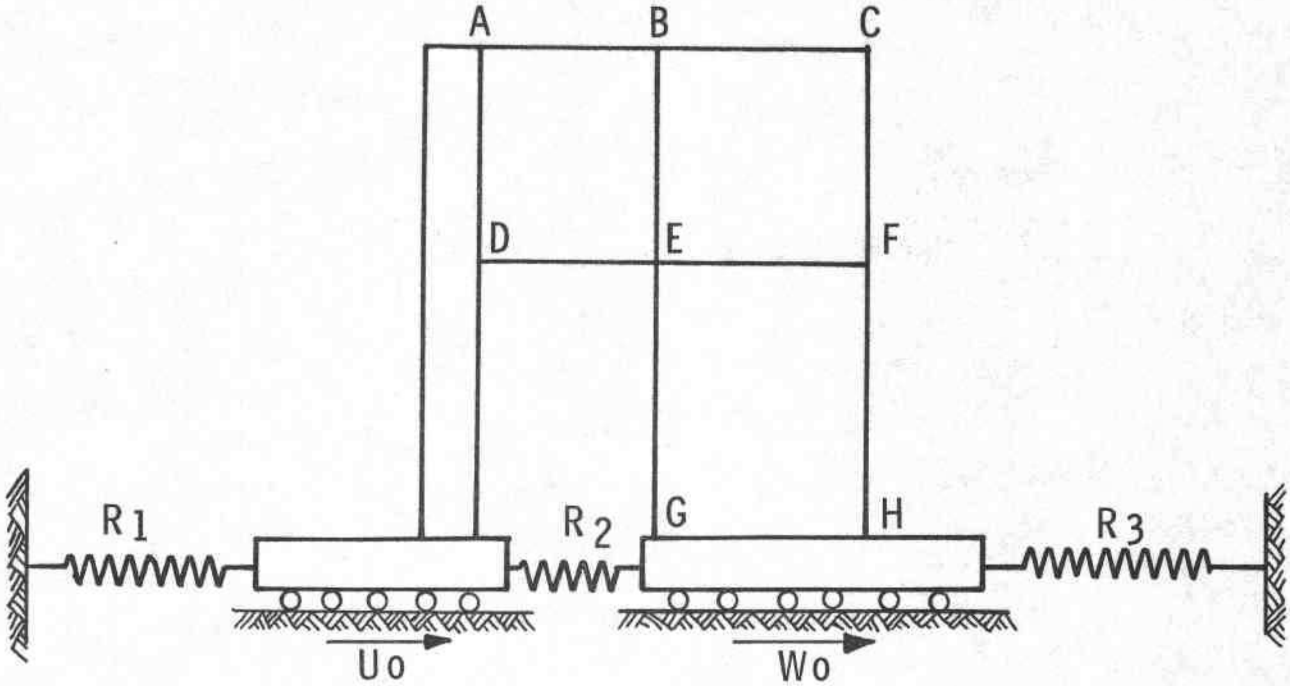

FIGURE 5

A COMPOSITE STRUCTURE WITH EXTRA DEGREES OF FREEDOM AT THE FOUNDATION LEVEL

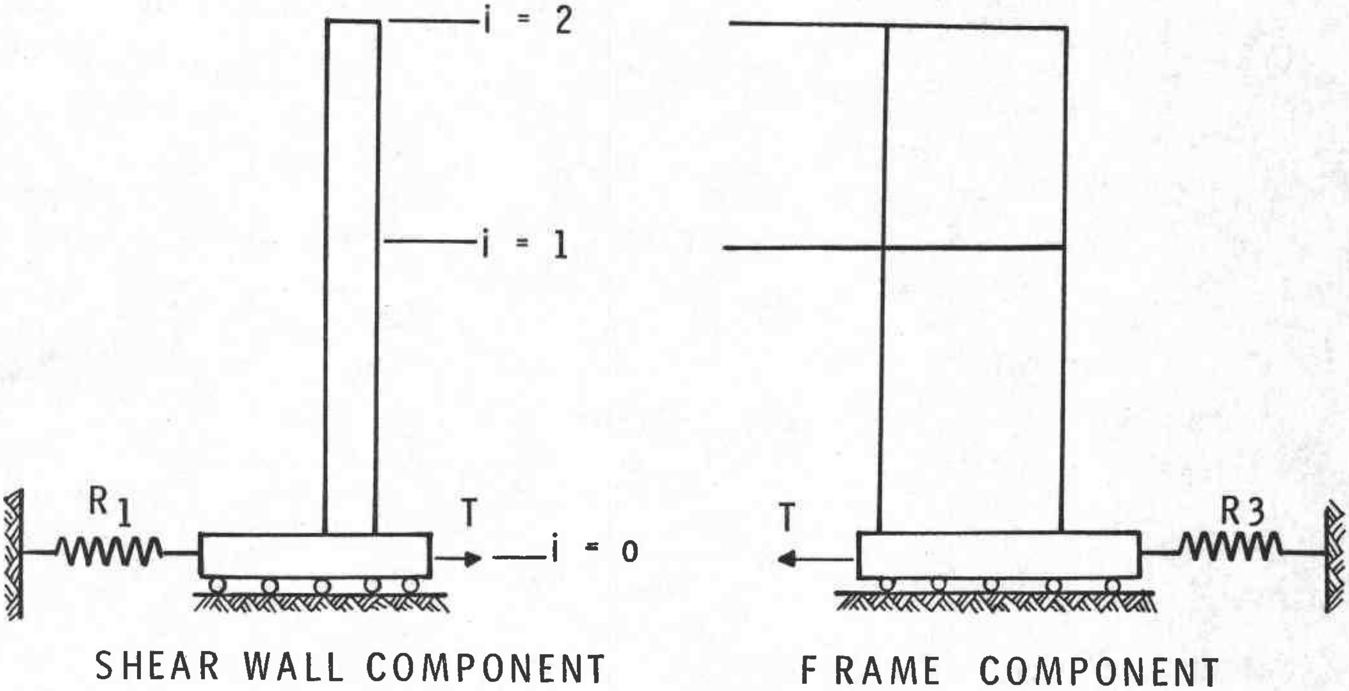

FIGURE 6

COMPONENT PARTS OF THE COMPOSITE STRUCTURE IN FIGURE 5

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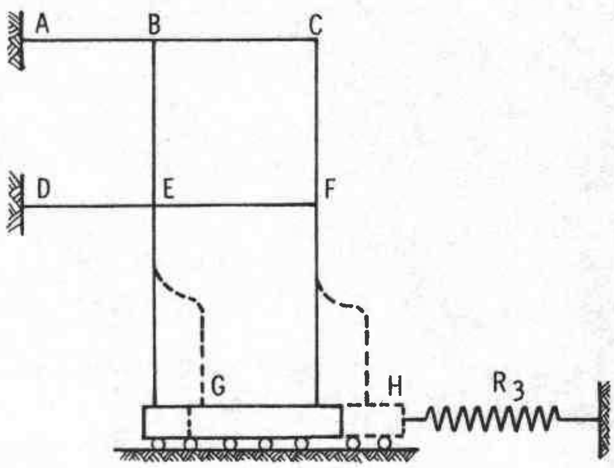

AB	BA	BE	BC	CB	CF
2.1		-12.5			-16.6
-1.0	4.2	4.1	4.2	8.3	8.3
		2.0	4.1	2.1	2.1
-2.0	-2.0	-2.0	-2.1	-2.1	-2.1
		0.5	-1.0	-1.0	-1.0
	0.5	0.5	0.5	1.0	1.0
1.1	2.7	8.4	5.7	8.3	-8.3

DE	ED	EB	EF	EA	FE	FC	FH
-12.5				100			100
	-25	-25	-25	-25	-33.3	33.3	-33.3
2.0			-16.6		-12.5		
	4.1	4.1	4.3	4.1	4.1	4.2	4.2
-0.5		2.0	2.0		2.1	4.1	
	-1.0	-1.0	-1.0	-1.0	-2.1	-2.1	-2.0
		-1.0	-1.0		-0.5	-1.0	
	0.5	0.5	0.5	0.5	0.5	0.5	0.5
-11.0	-21.4	-20.4	-36.8	78.6	-41.7	-27.6	69.4

GE	HF
100	100
-12.5	-16.6
2.0	2.1
-0.5	-1.0
89.0	84.5

A POSITIVE MOMENT OF $100 = \frac{6k_f}{h^2}$ AND ACTS IN A CLOCKWISE SENSE

$$r_{02} = \frac{3.8k_f}{h^3}$$

$$t_{02} = \frac{0.23k_f}{h^3}$$

$$x_{02} = \frac{0.07k_f}{h^2}$$

$$r_{01} = \frac{-23.18k_f}{h^3}$$

$$t_{01} = \frac{-1.94k_f}{h^3}$$

$$x_{01} = \frac{-0.66k_f}{h^2}$$

$$r_{00} = \frac{19.3k_f}{h^3} + R_3$$

FIGURE 7

MOMENT DISTRIBUTION ANALYSIS TO DETERMINE THE STIFFNESS COEFFICIENTS FOR THE CASE WHEN THE FOUNDATION OF THE FRAME COMPONENT MOVES UNIT DISTANCE

BA 4632-7

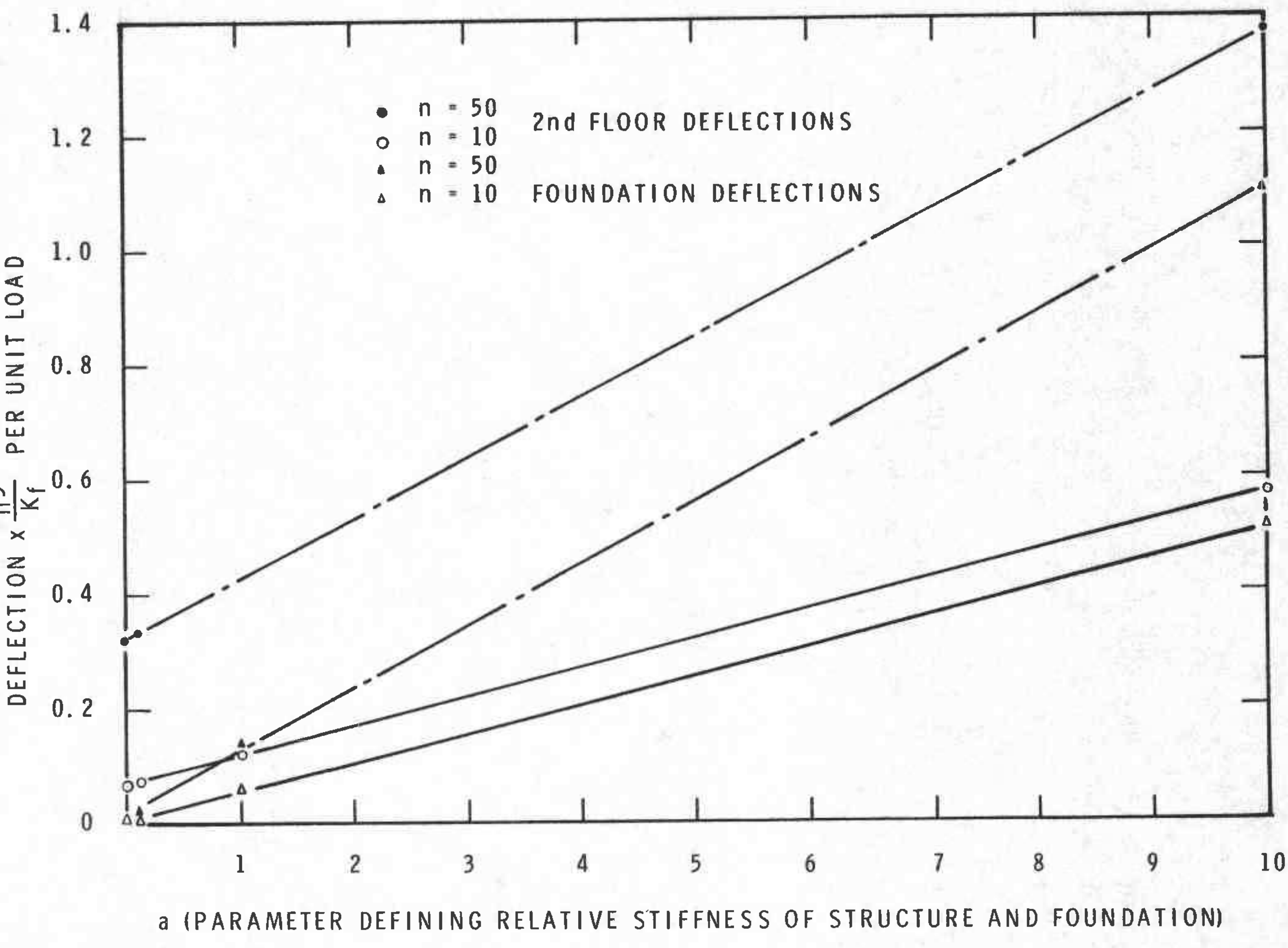

FIGURE 8 SOME OF THE FLEXIBILITY COEFFICIENTS FOR THE STRUCTURE IN FIGURE 5

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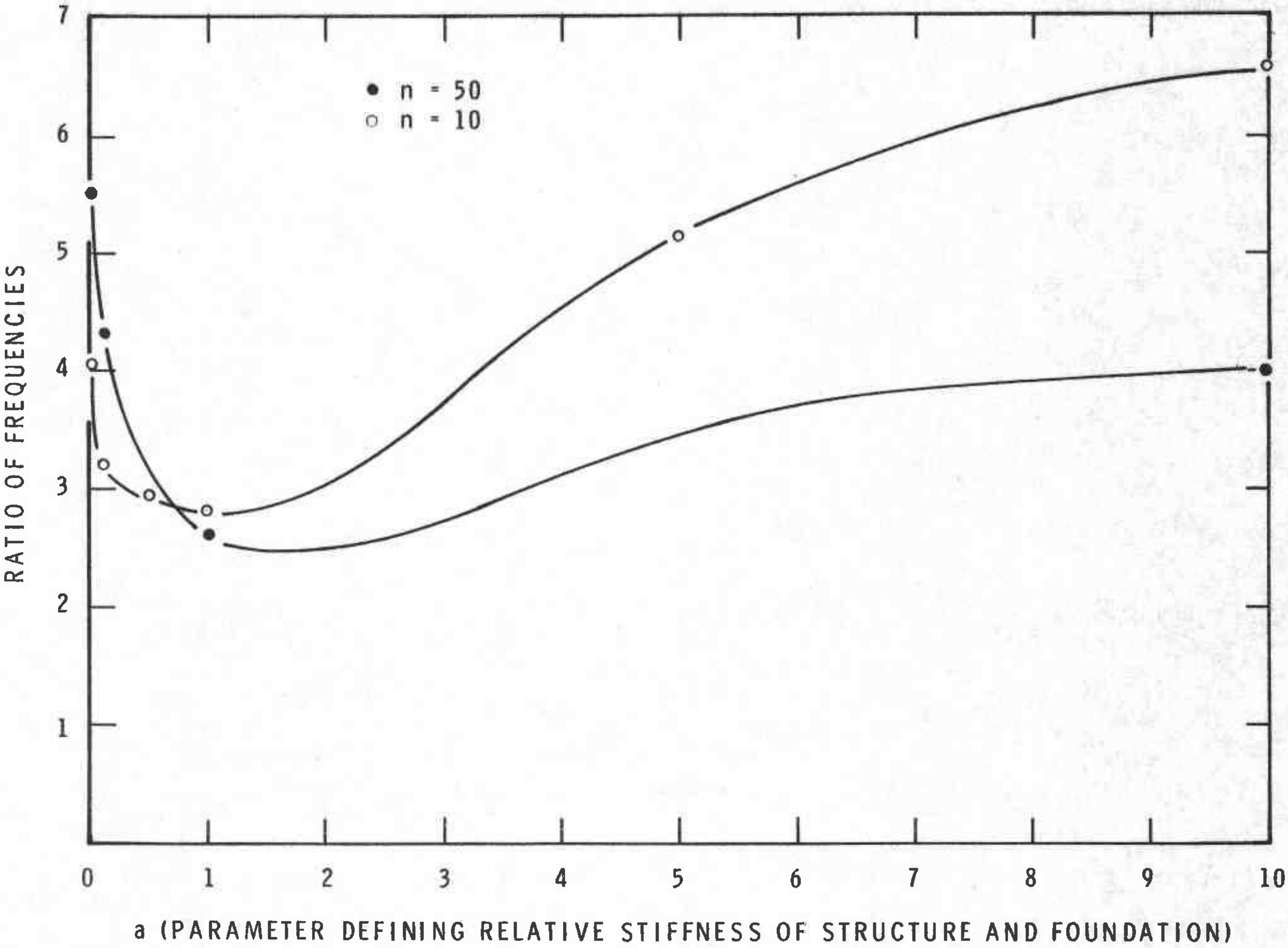

FIGURE 9

RATIO OF THE SECOND MODE FREQUENCY TO THE FUNDAMENTAL FREQUENCY FOR THE STRUCTURE IN FIGURE 5

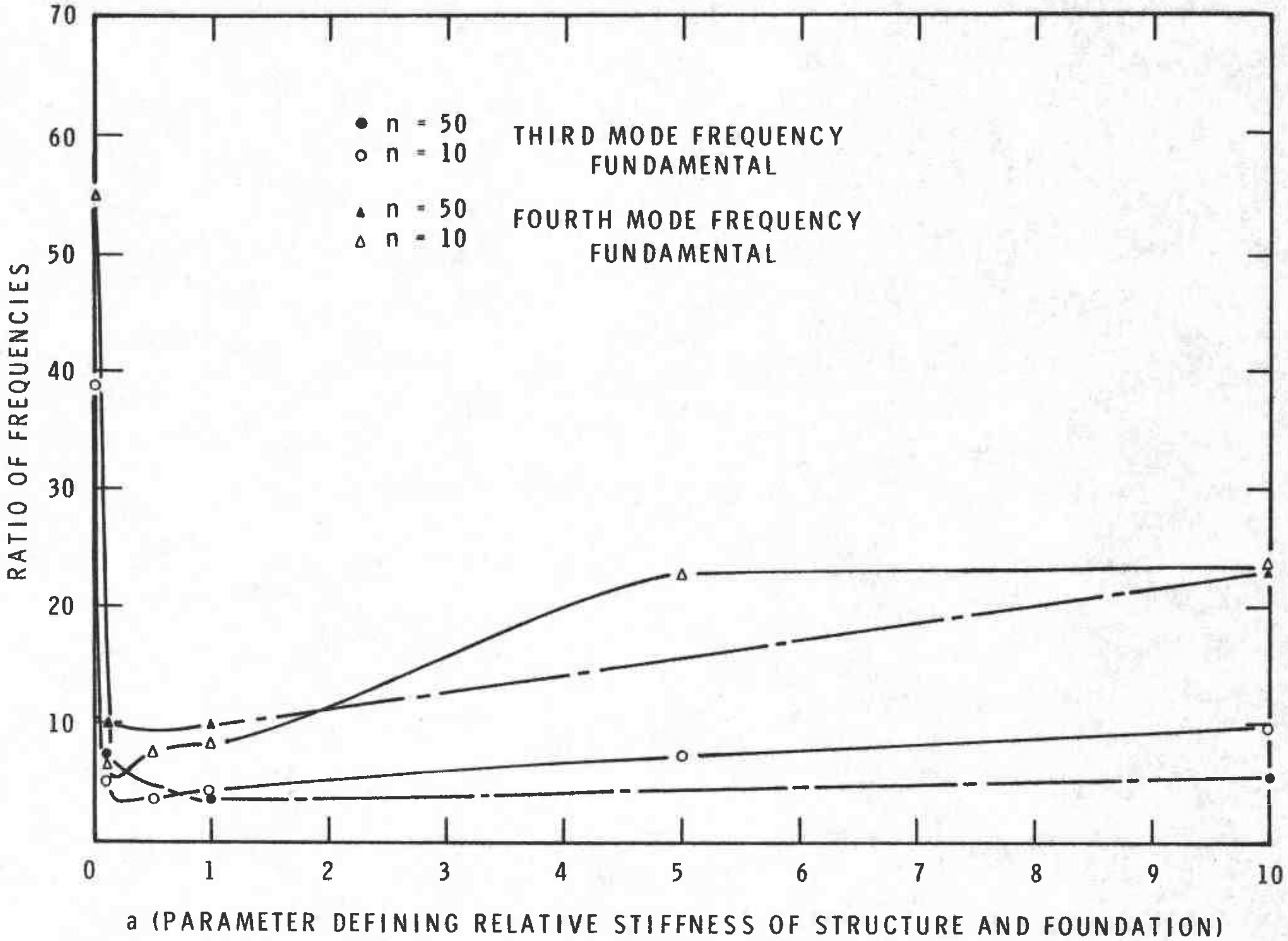

FIGURE 10

RATIO OF THE HIGHER MODE FREQUENCIES TO THE FUNDAMENTAL FOR THE STRUCTURE IN FIGURE 5

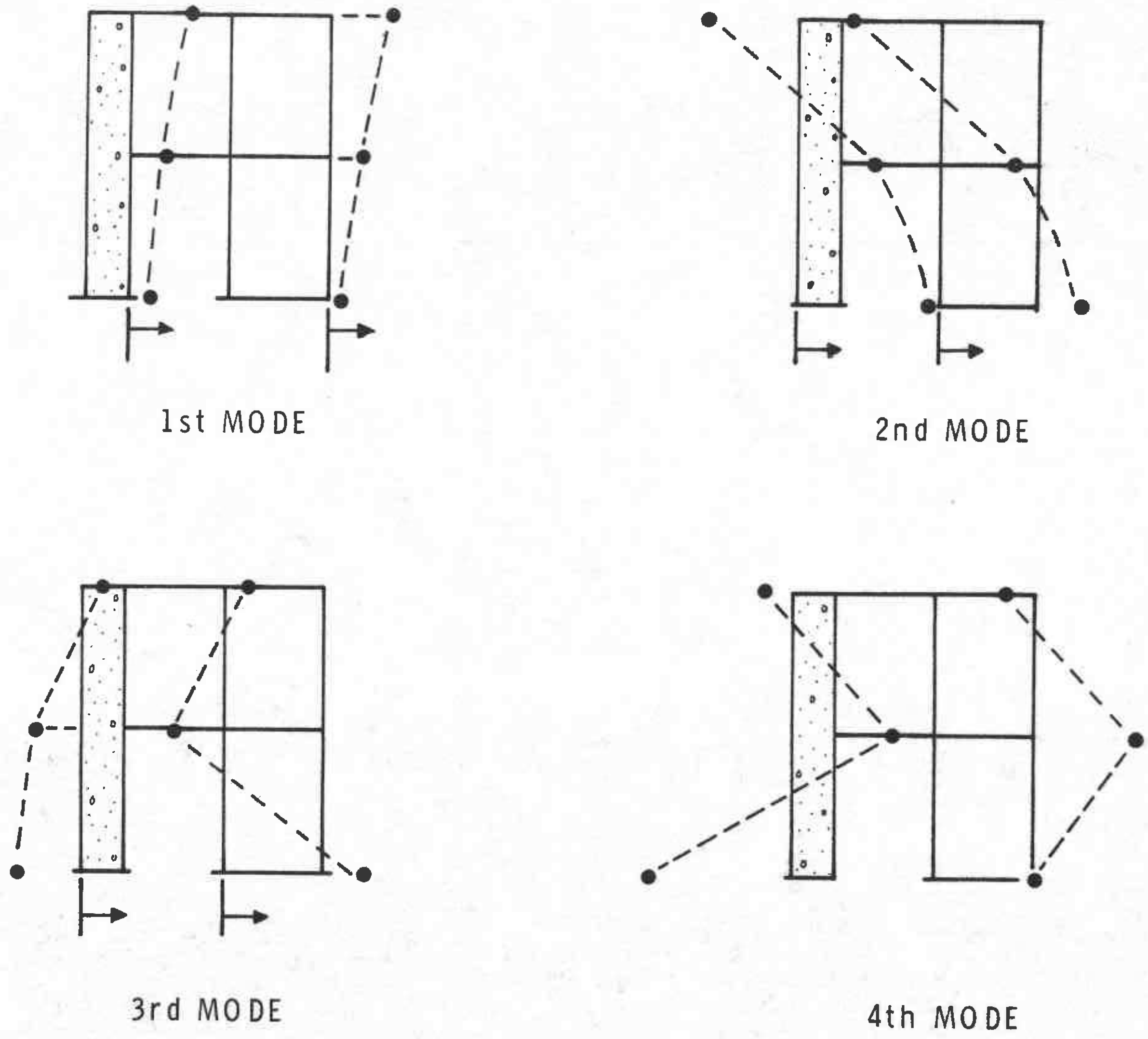

FIGURE 11

THE MODES SHAPES FOR THE STRUCTURE IN FIGURE 5 WHEN n = 50 AND a = 1

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