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Finite element and sensitivity analysis of thermally induced flow instabilities

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SUMMARY

This paper presents a finite element algorithm for the simulation of thermo-hydrodynamic instabilities causing manufacturing defects in injection molding of plastic and metal powder. Mold-filling parameters determine the flow pattern during filling, which in turn influences the quality of the final part. Insufficiently, well-controlled operating conditions may generate inhomogeneities, empty spaces or unusable parts. An understanding of the flow behavior will enable manufacturers to reduce or even eliminate defects and improve their competitiveness. This work presents a rigorous study using numerical simulation and sensitivity analysis. The problem is modeled by the Navier–Stokes equations, the energy equation and a generalized Newtonian viscosity model. The solution algorithm is applied to a simple flow in a symmetrical gate geometry. This problem exhibits both symmetrical and non-symmetrical solutions depending on the values taken by flow parameters. Under particular combinations of operating conditions, the flow was stable and symmetric, while some other combinations leading to large thermally induced viscosity gradients produce unstable and asymmetric flow. Based on the numerical results, a stability chart of the flow was established, identifying the boundaries between regions of stable and unstable flow in terms of the Graetz number (ratio of thermal conduction time to the convection time scale) and B , a dimensionless ratio indicating the sensitivity of viscosity to temperature changes. Sensitivities with respect to flow parameters are then computed using the *continuous sensitivity equations* method. We demonstrate that sensitivities are able to detect the transition between the stable and unstable flow regimes and correctly indicate how parameters should change in order to increase the stability of the flow. Copyright © 2009 John Wiley & Sons, Ltd.

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KEY WORDS: stable and unstable flows; generalized Newtonian fluids; stability chart; sensitivity analysis

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1. INTRODUCTION

Wall-bounded, pressure-driven flow of a fluid with temperature-dependent viscosity has the potential for thermally induced instability. For example, a small decrease in the temperature will cause an increase in viscosity that will in turn decrease the flow and allow further localized cooling. Under certain circumstances, these changes will cascade until the flow ceases in one region and accelerates in other regions. This kind of flow behavior can produce manufacturing defects during plastic and metal powder injection molding [1, 2] as shown in Figure 1. This paper focuses on a metal powder injection molding application for which specific processing conditions lead to thermally induced flow instabilities. The purpose of the work is to provide physical understanding of the flow behavior by a rigorous numerical study of its stability as a function of flow parameters and to establish general conditions under which the flow remains stable. The sensitivities of the solution with respect to flow parameters are also investigated and correlations are made between flow sensitivities and stability.

There is a wealth of work in the literature about injection molding. Depending on the problem, various types of mathematical models are proposed, ranging from simplified approaches [3–6] (i.e. Hele–Shaw approximation) to more complex models where flow and heat transfer equations are solved on 3D geometries [7–9]. A series of studies done by Ilinca and Hétu deal with 3D modeling of the filling and post-filling phases of the injection molding process [10], gas-assisted injection molding [11], co-injection molding [12] and injection of metal powders [1]. For a review of the research in mold filling simulation the reader may consult the work of Kim and Turng [13].

Works on the stability of non-Newtonian fluids go back to the late 60s and early 70s. Shah and Pearson [14–16] published a series of articles on the stability of polymer flows in a radial geometry. The latest one in the series [16] focuses on a power law model for the viscosity. This study is the first one to analyze and characterize the instability of thermoviscous flows in terms of the dimensionless numbers Graetz (Gz) and B . Inspired by this work, Stevenson *et al.* [1, 2] presented a stability analysis for a radial flow of metal powder during injection molding processes. They performed 3D simulations using a generalized Newtonian viscosity model and investigated the effect of inertia and yield stress. The numerical model was able to recover the non-uniform filling pattern observed for given operating conditions as shown in Figures 1(b) and (c). Costa and

Figure 1. (a) Thermal flow instability in an ABS plastic part for slow filling from a central gate; (b) uneven filling pattern for metal powder injection molding with radial gate; and (c) 3D simulation of observed flow instabilities.

Macedonio [17] use a similar analysis in the study of magmatic flows in volcanic conduits. Other articles and books [18, 19] may also be consulted.

Sensitivity analysis has a wide range of applications, from driving optimization, to flow control and fast evaluation of nearby solutions. The sensitivities equation method was first developed in solid mechanics [20, 21] and then expanded to other engineering fields, such as fluid mechanics. *Flow sensitivities* refer to the derivatives of the flow solution (velocity, temperature, pressure, etc.) with respect to a parameter a . For example, if the velocity $\mathbf{u}$ depends on space, time (i.e. $\mathbf{x}, t$) and a parameter a , then its sensitivity with respect to a is usually written as:

$$s_{\mathbf{u}}^a = \frac{\partial \mathbf{u}(\mathbf{x}, t; a)}{\partial a} \quad (1)$$

The sensitivity of $\mathbf{u}$ expresses how the velocity field responds to perturbations of a around its nominal value. In other words, it measures the importance of changes in the flow response to perturbations of the model parameters. There are several means of computing sensitivities [22–24]. In this work, we have adopted the *continuous sensitivity equation* (CSE) method [25–27]. The CSEs are derived formally by implicit differentiation of flow equations with respect to parameter a . The sensitivity equations are then discretized and solved numerically to obtain the desired sensitivities. Aerodynamics applications of the CSE method may be found in Godfrey and Cliff [28, 29], Borggaard and Burns [25], Limache [30] and Turgeon *et al.* [31]. Application to heat conduction is reported by Blackwell *et al.* [32]. Sensitivities for incompressible flows with heat transfer may be found in several references [33–35]. Solution of the sensitivity equations for transient flow of non-Newtonian fluids is presented by Ilinca *et al.* [36]. Application of CSE to liquid composite molding is reported by Henz *et al.* [37, 38].

This paper presents a detailed flow and sensitivity analysis for a generalized Newtonian fluid flowing inside a T -shaped geometry. This flow was chosen as a 2D analog of the circular gate used in the earlier work of Ilinca *et al.* [1, 2] with the added simplification that we assume the domain is filled with fluid initially. This avoids complication resulting from front tracking during mold filling, yet it retains all the elements of interest, namely thermally induced flow instability. The model problem and flow equations are presented in Section 2. The solution algorithm is briefly described in Section 3, followed by the presentation of the numerical results. The paper ends with conclusions.

2. MATHEMATICAL MODEL

For injection molding applications the mold-filling problem involves the solution of the evolving free surface inside the mold cavity. However, thermally induced flow instabilities are triggered inside the mold gate (smaller section by which the material enters the mold cavity) and are observed generally when the gate is entirely filled. Hence, in the present work we focus only on the flow inside the gate, which is considered filled from the start of the simulation and no free surface solution is needed. In this section, the mathematical model is presented along with the geometry of the computational domain and boundary conditions. Flow equations are also given in a non-dimensional form and the flow parameters are identified. Then, the CSEs are presented.

2.1. Flow equations

The governing equations for a laminar incompressible flow are the continuity and momentum equations

$$\nabla \cdot \mathbf{u} = 0 \quad (2)$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \nabla \cdot [2\eta \dot{\boldsymbol{\gamma}}(\mathbf{u})] + \mathbf{f} \quad (3)$$

where ρ is the density, $\mathbf{u}$ is the velocity, p is the pressure, η is the viscosity, t represents the time, $\dot{\boldsymbol{\gamma}}(\mathbf{u}) = (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)/2$ is the shear rate tensor and $\mathbf{f}$ is a body force. Heat transfer is modeled by the energy equation as

$$\rho c_p \left(\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T \right) = \nabla \cdot (k \nabla T) + q_s + \Phi \quad (4)$$

where c_p is the specific heat, T is the temperature, k represents the heat conductivity, q_s is a heat source and Φ represents the viscous heating. The above system is closed with a proper set of initial conditions

$$\mathbf{u}(\mathbf{x}, t=0) = \mathbf{u}_0(\mathbf{x}) \quad \text{in } \Omega \quad (5)$$

$$T(\mathbf{x}, t=0) = T_0(\mathbf{x}) \quad \text{in } \Omega \quad (6)$$

and with proper Dirichlet and Neumann boundary conditions

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on } \Gamma_D^{\mathbf{u}} \quad (7)$$

$$(-p\mathbf{I} + 2\eta\dot{\boldsymbol{\gamma}}(\mathbf{u})) \cdot \hat{\mathbf{n}} = \bar{\mathbf{t}} \quad \text{on } \Gamma_N^t \quad (8)$$

$$T = \bar{T} \quad \text{on } \Gamma_D^T \quad (9)$$

$$k \nabla T \cdot \hat{\mathbf{n}} = \bar{q} \quad \text{on } \Gamma_N^q \quad (10)$$

where $\bar{\mathbf{u}}$ is the value of the velocity imposed along the boundary $\Gamma_D^{\mathbf{u}}$, $\mathbf{I}$ represents the identity tensor and $\bar{\mathbf{t}}$ is the imposed boundary distribution of the traction force. In our application, $\mathbf{f} = 0$, $q_s = 0$ and Φ are neglected.

2.2. Viscosity model

The rheological behavior of the fluid (i.e. plastic or metallic powders) is expressed mathematically by a non-linear relationship between the stress tensor $\boldsymbol{\tau}(\mathbf{u})$ and the shear rate tensor $\dot{\boldsymbol{\gamma}}$. Experimental observations indicate that this relationship is well described by a *generalized Newtonian fluid* model having the following constitutive equation:

$$\boldsymbol{\tau}(\mathbf{u}) = 2\eta(\dot{\boldsymbol{\gamma}}, T)\dot{\boldsymbol{\gamma}} \quad (11)$$

where $\dot{\boldsymbol{\gamma}}$ is the shear rate defined as

$$\dot{\boldsymbol{\gamma}} = \sqrt{2\dot{\gamma}_{ij}\dot{\gamma}_{ij}} = \sqrt{2\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2} \quad (12)$$

There are several empirical relationships between the viscosity and the shear rate. Among them, the Cross model, the Carreau model and the Ostwald power law model have been widely used [2, 10, 39]. We have chosen the following power law model for its simplicity and range of applications in heat transfer problems

$$\eta(\dot{\gamma}, T) = C\dot{\gamma}^m e^{-bT} \quad (13)$$

where C , m and b are model constants.

2.3. Geometry of the computational domain

The computational domain used for numerical simulations is derived from the original geometry used by Stevenson and Ilinca [2], a diaphragm gate with radial symmetry (see Figure 2(a)). This gate is designed to provide an even distribution of the fluid. The fluid enters at the bottom of the injector, flows upward, then flows between the two parallel plates and finally exits the injector to fill the mold cavity. The radial symmetry would normally allow for axisymmetric simulations. However, because the flow inside the gate may be unstable, a flow stability study would have required a full 3D model. For the present study, we use a 2D version shown in Figure 2(b). While the onset of instabilities may differ between a 2D and a 3D solution, the present 2D numerical model recovers the same behavior in terms of the flow dependence upon the temperature as the 3D counterpart. By choosing to solve on a 2D geometry, we are able to investigate more carefully the link between the solution stability and flow sensitivities. Dimensions of the present model are such as $r/H=0.5$, $R/H=3.5$ and $L/H=5$. The fluid enters through the bottom of the T with a steady velocity profile and a prescribed temperature T_a . The vertical walls are insulated, whereas the horizontal ones are maintained at a given temperature T_r . On the lower horizontal wall on the right-hand side of the gate, the temperature boundary condition is perturbed by 10^{-5} of the temperature difference between T_a and T_r in order to produce a slightly asymmetric temperature condition. This perturbation, which is marginally higher than the round-off errors, has no effect

Figure 2. (a) Schematic of an axisymmetric radial injector and (b) actual 2D computational domain with boundary conditions, $L=5.0$, $R=3.5$, $r=0.5$ and $H=1.0$.

on the stability of the flow and is used to control which side the unstable flow will favor. Finally, at the exits we set $v=0$ with free boundary conditions for u and T .

2.4. Velocity profile

A steady-state velocity profile was imposed at the inflow. Because the fluid is non-Newtonian with the viscosity described by a power law model, the velocity profile will not be the usual parabolic Newtonian flow. The appropriate velocity profile is obtained by solving Equation (3) for v in the vertical channel of the T -shaped geometry. The solution satisfies $v(x=r)=v(x=-r)=0$, $\partial_x v(x=0)=0$ and the temperature is constant. We obtain the following velocity profile at the inlet:

$$v(x) = \frac{Q}{2r} \left(\frac{2m+3}{m+2} \right) \left[1 - \left| \frac{x}{r} \right|^{(m+2)/(m+1)} \right] \quad (14)$$

where Q is the inlet flow rate. Note that we have the familiar Newtonian velocity profile when $m=0$. Here, we have set the value of m to -0.8 so that both stable and unstable flow patterns may be observed depending on the flow parameters [15]. Thus, $v(x)$ is given by

$$v(x) = \left(\frac{7Q}{12r} \right) \left[1 - \left(\frac{x}{r} \right)^6 \right] \quad (15)$$

2.5. Dimensionless equations

Equations (2)–(4) may be written in dimensionless form by choosing proper scales for the variables and parameters of the problem. These reference values are identified by the subscript 0. In order to preserve the original form of Equations (2)–(4) and simplify code implementation, we have chosen an *isotropic scale* to obtain our dimensionless equations. The length scale is taken equal to H , the half-width of the gate and the velocity scale is set to the mean velocity in the horizontal channel $u_0 = Q/(4H)$. Therefore, we have

$$\tilde{x} = \frac{x}{H}, \quad \tilde{y} = \frac{y}{H}, \quad \frac{\partial}{\partial x} = \frac{1}{H} \frac{\partial}{\partial \tilde{x}} \quad \text{and} \quad \frac{\partial}{\partial y} = \frac{1}{H} \frac{\partial}{\partial \tilde{y}} \quad (16)$$

where the $\sim$ symbol defines the non-dimensional variables and properties. For the velocity and time we have

$$\tilde{u} = \frac{u}{u_0}, \quad \tilde{v} = \frac{v}{u_0} \quad \text{and} \quad \tilde{t} = \frac{t}{t_0} \quad \text{where} \quad t_0 = \frac{H}{u_0} \quad (17)$$

The dimensionless pressure is defined as $\tilde{p} = p/p_0$, where the viscous pressure scale p_0 is given by

$$p_0 = \frac{\eta_0 u_0}{H} \quad (18)$$

The dimensionless temperature is defined as

$$\tilde{T} = \frac{T - T_r}{\Delta T_0} \quad (19)$$

where T_r is the reference temperature taken equal to the gate wall temperature. ΔT_0 is the temperature scale given by $T_a - T_r$, where T_a represents the fluid temperature at gate inlet. In the same manner, the dimensionless properties are:

$$\tilde{\rho} = \frac{\rho}{\rho_0}, \quad \tilde{c}_p = \frac{c_p}{c_{p0}}, \quad \tilde{k} = \frac{k}{k_0} \quad \text{and} \quad \tilde{\eta} = \frac{\eta}{\eta_0} \quad (20)$$

The density, specific heat and thermal conductivity are considered as constant and equal to their respective scale values. Therefore, $\tilde{\rho}$, $\tilde{c}_p$ and $\tilde{k}$ are equal to unity. The viscosity scale η_0 is computed from

$$\eta_0 = C \dot{\gamma}_0^m e^{-bT_r} \quad \text{where} \quad \dot{\gamma}_0 = \frac{u_0}{H} \quad (21)$$

The viscosity dependence on the temperature could be quantified by introducing the dimensionless number $B = \Delta T_0 / \Delta T_{\text{rheol}}$, where $\Delta T_{\text{rheol}} = 1/b$. The characteristic temperature ΔT_{rheol} is the temperature increase (or decrease) that is needed to obtain a decrease (or increase) in the viscosity by a factor of e , the natural logarithmic basis. Finally, the dimensionless viscosity is given by

$$\tilde{\eta} = \tilde{\gamma}^m e^{-B\tilde{T}} \quad (22)$$

By substituting Equations (16)–(22) into (2)–(4), we obtain the dimensionless continuity, momentum and energy equations.

$$\tilde{\nabla} \cdot \tilde{\mathbf{u}} = 0 \quad (23)$$

$$Re \left(\frac{\partial \tilde{\mathbf{u}}}{\partial \tilde{t}} + \tilde{\mathbf{u}} \cdot \tilde{\nabla} \tilde{\mathbf{u}} \right) = -\tilde{\nabla} \tilde{p} + \tilde{\nabla} \cdot [\tilde{\eta} (\tilde{\nabla} \tilde{\mathbf{u}} + (\tilde{\nabla} \tilde{\mathbf{u}})^T)] \quad (24)$$

$$Re Pr \left(\frac{\partial \tilde{T}}{\partial \tilde{t}} + \tilde{\mathbf{u}} \cdot \tilde{\nabla} \tilde{T} \right) = \tilde{\nabla} \cdot \tilde{\nabla} \tilde{T} \quad (25)$$

Note that Equations (23)–(25) preserve the original form of the continuity, momentum and energy equations (2)–(4). In the above equations $Re = \rho_0 u_0 H / \eta_0$ is the *Reynolds* number and $Pr = \rho_0 c_{p0} / k_0$ the *Prandtl* number. We can also write the energy equation in terms of the *Graetz* number, which is the ratio between the time scale of the heat transfer by conduction in the direction normal to the cooled walls ($t_{\text{cond}} = \rho_0 c_{p0} H^2 / k_0$) and the time scale of the heat transfer by convection in the direction of the flow ($t_{\text{conv}} = u_0 / R$):

$$Gz = \left(\frac{\rho_0 c_{p0} u_0 H^2}{k_0 R} \right) = Re Pr \left(\frac{H}{R} \right) \quad (26)$$

In terms of the Gz number, the energy equation takes the form

$$\frac{\partial \tilde{T}}{\partial \tilde{t}} + \tilde{\mathbf{u}} \cdot \tilde{\nabla} \tilde{T} = \tilde{\nabla} \cdot \left[\frac{1}{Gz} \left(\frac{H}{R} \right) \tilde{\nabla} \tilde{T} \right] \quad (27)$$

Materials used in the applications of interest have high viscosity so that the inertial terms are small when compared with the viscous forces. Therefore, Re is small compared with unity and we can drop the inertial term in the momentum equation (24). By dropping the $\sim$ over the

non-dimensional variables, we finally obtain

$$0 = \nabla \cdot \mathbf{u} \quad (28)$$

$$0 = -\nabla p + \nabla \cdot [\eta(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)] \quad (29)$$

$$\left(\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T \right) = \nabla \cdot \left[\frac{1}{Gz} \left(\frac{H}{R} \right) \nabla T \right] \quad (30)$$

$$\eta = \dot{\gamma}^m e^{-BT} \quad (31)$$

Notice that except for a given ratio H/R , the above equations depend only on two dimensionless numbers: Gz and B .

2.6. Sensitivity equations

The CSEs are derived formally by implicit differentiation of Equations (28)–(30) with respect to a given parameter a . We treat the dependent variables (i.e. $\mathbf{u}$, p and T) as functions of space, time and a . This dependence is denoted by

$$\mathbf{u} = \mathbf{u}(\mathbf{x}, t; a), \quad p = p(\mathbf{x}, t; a) \quad \text{and} \quad T = T(\mathbf{x}, t; a) \quad (32)$$

The sensitivities for the dependent variables are then

$$s_{\mathbf{u}} = \frac{\partial \mathbf{u}}{\partial a}, \quad s_p = \frac{\partial p}{\partial a} \quad \text{and} \quad s_T = \frac{\partial T}{\partial a} \quad (33)$$

By denoting the total derivatives of fluid properties and other flow parameters by d/da , the system of sensitivity equations is given by:

$$0 = \nabla \cdot s_{\mathbf{u}} \quad (34)$$

$$0 = -\nabla s_p + \nabla \cdot [\eta(\nabla s_{\mathbf{u}} + (\nabla s_{\mathbf{u}})^T)] + \nabla \cdot \left[\frac{d\eta}{da} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) \right] \quad (35)$$

$$\frac{\partial s_T}{\partial t} + \mathbf{u} \cdot \nabla s_T + s_{\mathbf{u}} \cdot \nabla T = \nabla \cdot \left[\frac{1}{Gz} \frac{H}{R} \nabla s_T \right] - \nabla \cdot \left[\frac{1}{Gz^2} \frac{H}{R} \left(\frac{dGz}{da} \right) \nabla T \right] \quad (36)$$

Because the viscosity depends on the shear rate and temperature

$$\eta = \eta(\dot{\gamma}(\mathbf{u}(a)), T(a); a) \quad (37)$$

its sensitivity is given by:

$$\frac{d\eta}{da} = \frac{\partial \eta}{\partial \dot{\gamma}} \frac{\partial \dot{\gamma}}{\partial \mathbf{u}} s_{\mathbf{u}} + \frac{\partial \eta}{\partial T} s_T + \frac{\partial \eta}{\partial a} \quad (38)$$

Initial conditions for the sensitivity equations are obtained by implicit differentiation of Equations (5) and (6) as

$$s_{\mathbf{u}}(\mathbf{x}, t=0) = \frac{\partial \mathbf{u}_0}{\partial a}(\mathbf{x}) \quad \text{in } \Omega \quad (39)$$

$$s_T(\mathbf{x}, t=0) = \frac{\partial T_0}{\partial a}(\mathbf{x}) \quad \text{in } \Omega \quad (40)$$

whereas Dirichlet and Neumann boundary conditions are obtained in a similar manner from Equations (7)–(10) as

$$s_{\mathbf{u}} = \frac{\partial \bar{\mathbf{u}}}{\partial a} \quad \text{on } \Gamma_{\mathbf{D}}^{\mathbf{u}} \quad (41)$$

$$\left(-s_p \mathbf{I} + 2 \frac{d\eta}{da} \dot{\gamma}(\mathbf{u}) + 2\eta \dot{\gamma}(s_{\mathbf{u}}) \right) \cdot \hat{\mathbf{n}} = \frac{\partial \bar{t}}{\partial a} \quad \text{on } \Gamma_{\mathbf{N}}^t \quad (42)$$

$$s_T = \frac{\partial \bar{T}}{\partial a} \quad \text{on } \Gamma_{\mathbf{D}}^T \quad (43)$$

$$\left(\frac{dk}{da} \nabla T + k \nabla s_T \right) \cdot \hat{\mathbf{n}} = \frac{\partial \bar{q}}{\partial a} \quad \text{on } \Gamma_{\mathbf{N}}^q \quad (44)$$

3. SOLUTION METHOD

3.1. Finite element solution

The flow equations are solved by a finite element method on 2D meshes. Velocity and temperature are discretized using second-order interpolation functions, whereas the pressure is discretized with linear functions (*P2–P1* Taylor–Hood triangular element). The momentum–continuity and energy equations are solved by a Galerkin finite element method [40]. The energy equation is dominated by convection and the Galerkin method could lead to unphysical oscillations. Hence, simulations were also carried out using an SUPG stabilized method and the results were found to be similar to those by the Galerkin method. This shows that the Galerkin method using quadratic interpolation functions and reffined meshes, as is the case in the present work, works well and that there is no need for a stabilized finite element method. Sensitivity equations are discretized using the same Galerkin finite element formulation as for the flow equations. In theory, the CSE can be solved by any numerical method. In practice, it is convenient and cost effective to use the same finite element method for the flow and the CSE. Indeed, we note that the CSE amounts to a Newton linearization of the Navier–Stokes equations. Thus, if one uses Newton’s method for solving the finite element equations for the flow, the flow and sensitivity equations will have the same finite element matrix. Only the right-hand side will differ. This results in substantial savings since the matrix of sensitivities need not be recomputed. Finally, time is discretized by an implicit Euler scheme.

3.2. Implementation

The solution algorithm works as follows: At each time step

- iterate over the non-linear momentum–continuity and energy equations (28)–(30) until convergence. A few steps of successive substitution (Picard’s iteration) are performed at the beginning of the first time step and the Newton method is used afterward;
- use the matrix from the last Newton iteration on the flow problem and solve the linear system for the sensitivities equations (34)–(36).

The linearized equations are assembled in a sparse matrix. The resulting systems of linear algebraic equations for the flow and sensitivities are then solved by LU (Gaussian) factorization.

4. NUMERICAL RESULTS

In this section, the numerical approach is first verified using the method of manufactured solution (MMS). A grid and time-step refinement study is performed to assess the grid convergence and accuracy of the flow and sensitivity solutions. This verification exercise is carried out for a generalized Newtonian flow and sensitivities are computed with respect to the flow parameters. Next, we present the results obtained for the flow inside a 2D injector presenting thermally induced flow instabilities.

4.1. Verification

Before proceeding with simulations one must ensure that the code is properly implemented and delivers the expected accuracy. Following the philosophy of Boehm, Blotter and Roache, the three most important steps are: *Code Verification*, *Verification of Calculations* and *Validation* [41, 42]. Verification of a code involves error evaluation from a known solution to establish that the CFD code works correctly. Verification of a calculation involves error estimation. Both verification steps are purely numerical exercises with no concern whatsoever for the realism of the physical laws used in the code. Finally, validation is concerned with the agreement of the mathematical model with the physical system of interest. In other words:

- ‘*Verification*’ $\sim$ solving the equations *right*.
- ‘*Validation*’ $\sim$ solving the *right* equations.

4.1.1. Method of manufactured solutions. MMS provides a general procedure for generating analytical solutions for code verification. The procedure is very simple. We first pick a continuum solution that will generally not satisfy the governing equations, because of the arbitrary nature of our choice. The solution should be non-trivial in the sense that it exercises all derivatives in the PDE. An appropriate source term is generated to cancel out any imbalance in the original PDE caused by the choice of the analytical solution. In this work, we use the following analytical solution inspired from an exact solution of the 2D Navier–Stokes equations obtained by Taylor [43]. Velocity components u and v , temperature T and pressure p are given by

$$u(x, y) = Gz^2 m^2 \sin(\pi x) \cos(\pi y) e^{(-B\omega t)} \quad (45)$$

$$v(x, y) = -Gz^2 m^2 \sin(\pi y) \cos(\pi x) e^{(-B\omega t)} \quad (46)$$

$$T(x, y) = \frac{Gz^2 m^2}{4} (1 + \sin^2(\pi x) \cos^2(\pi y)) e^{(-B\omega t)} \quad (47)$$

$$p(x, y) = Gz^2 m^2 (\cos(\pi x) + \sin(\pi y)) e^{(-B\omega t)} \quad (48)$$

where Gz , B and m are the parameters in Equations (28)–(31), t denotes time and ω is an arbitrary selected parameter. The viscosity is given by a power law model and the parameters take on the values typical of the problem of interest: $Gz=2$, $B=2$, $m=-0.8$ and $\omega=0.1$. The various analytical expressions for the sensitivities of u , v , T and p with respect to Gz , B and m

are obtained by direct differentiation of Equations (45)–(48). The computational domain for the flow and sensitivities and their respective boundary conditions is presented in Figure 3. Dirichlet boundary conditions are imposed for u , v and T from the exact solution on the domain bottom and vertical boundaries. On the top boundary, Dirichlet boundary conditions are imposed for u and T , whereas the pressure level is imposed through a Neumann boundary condition for v .

4.1.2. Convergence analysis. A grid and time-step convergence study is carried out for the flow and its sensitivities with respect to the following three parameters: Gz , B and m . Because the analytical solutions for the flow and their sensitivities are known, we can determine the true error E_{true} by evaluating the difference between the finite element solution u_h and the analytical solution u_{exa} , that is: $E_{\text{true}} = u_h - u_{\text{exa}}$. By performing grid and time-step refinement, the convergence rate of the error can be determined and compared with the *a priori* rate of convergence of the finite element method. The mesh and time steps for the grid sequence are presented in Table I. The space–time interpolation scheme is accurate to $\mathcal{O}(\delta t)$ in time and $\mathcal{O}(\delta x^2)$ in space. If we choose to refine the element size h by 2, the spatial error decreases by 4, thus we must divide Δt by 4 to also decrease the temporal error by a factor of 4 from one mesh to the next. Figure 4 shows examples

Figure 3. Verification problem: computational domain with $H = L = 2$.

Table I. Mesh size h and time steps Δt for verification problem.

Mesh	h	Δt
1	0.5	$\frac{1}{10}$
2	0.25	$\frac{1}{40}$
3	0.125	$\frac{1}{160}$
4	0.0625	$\frac{1}{640}$
5	0.03125	$\frac{1}{2560}$
6	0.015625	$\frac{1}{10240}$

Figure 4. Examples of meshes used for the verification problem: (a) 8×8 elements mesh and (b) 32×32 elements mesh.

Figure 5. Verification problem: (a) convergence of the norm of the velocity field; (b) and its sensitivities with respect to Gz ; (c) m ; and (d) B .

of meshes used to perform our verification. Structured and regular meshes are well adapted for verification problems, since we have optimum and precise control over the element size h .

Figures 5 and 6 show the results for the convergence of the error for the H_1 semi-norm of the velocity and temperature fields and of their sensitivities with respect to the parameters Gz ,

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Figure 6. Verification problem: (a) convergence of the norm of the temperature field; (b) and its sensitivities with respect to Gz ; (c) m ; and (d) B .

Table II. Verification problem: convergence rates for the flow and its sensitivities.

Solutions norm	Flow	Sensitivity parameter Gz	Sensitivity parameter m	Sensitivity parameter B
Velocity field	1.9914	2.0896	2.0102	1.9898
Temperature field	1.9837	2.0017	1.9862	1.9856

B and m . In addition to the true error, an estimated error is calculated by taking the difference between the finite element solution u_h and $\tilde{u}$, an approximation of the true solution. Error estimates are obtained by a local least-square reconstruction of the solution derivatives as proposed by Zienkiewicz and Zhu [44, 45]. As can be seen, the curves for the true and estimated errors for the flow and its sensitivities decrease with each mesh refinement. Table II shows that the convergence rates for the true error are in very good agreement with the predicted value given by the finite element method. Moreover, the error estimates converge at the same rate as and toward the true error. Therefore, we can say that the code is verified and that we are solving the equations *right*.

4.2. Transient flow of a generalized Newtonian fluid

4.2.1. Problem statement. In this work, we wish to establish the flow behavior of a generalized Newtonian fluid inside the symmetric injector described in Section 2.3. We present the numerical results obtained for several values of the parameters Gz and B . The value of m in the power law model for the viscosity is held fixed and taken equal to -0.8 . For this case the flow is expected to exhibit both stable and unstable patterns depending on the values of Gz and B . Recall that for $m=0$ the viscosity does not depend on the shear rate, but only on the temperature. In this case numerical tests indicate that the flow is stable for all the combinations of the Gz and B parameters.

The computational domain and boundary conditions for this problem are shown in Figure 2(b). Initial conditions for this problem are: $u=v=0$ for the velocity components and $T=1$ for temperature. For some values of Gz and B the flow is stable and symmetric while for other values, the flow is unstable and asymmetric or in transition between a stable and an unstable flow. Fluid enters the injector at constant speed with a uniform temperature. It goes through the injector up to the gate inlet where it separates into two streams and finally leaves the injector by the two opposed exits. The fluid cools down as it proceeds through the injector. Ideally, for a symmetric geometry and symmetric boundary conditions the flow will be symmetrical and the fluid will split evenly between the two exits. In reality, temperature imbalance may appear if thermal perturbations are present on the injector walls. Numerical simulations presented in this section show that a fluid with a shear rate and temperature-dependent viscosity has the potential to develop thermally induced flow instability. For example, a small decrease in temperature will cause an increase in viscosity that will in turn decrease flow and allow further localized cooling. For certain combinations of Gz and B numbers, these changes will cascade until the flow leaves the injector by only one exit.

4.2.2. The mesh. The flow behavior could be altered by physical and numerical perturbations, the mesh being a factor that may cause oscillations and artificial instabilities. Discretization errors can only be limited by imposing small convergence tolerances for the Newton iterations. Even very tiny mesh imperfections can trigger instabilities and transition toward non-symmetric flow patterns. Such instabilities cannot be the object of a rigorous sensitivity study because discretization errors cannot be controlled and used as a design parameter. Hence, we seek to minimize this kind of perturbations by using regular and symmetric meshes. The mesh used for the numerical simulations is similar to the one presented in Figure 7, except that it contains 74 561 nodes to ensure the accuracy of the numerical solutions.

4.2.3. Temperature imbalance and flow classification. In this work we use the temperature imbalance to characterize the different flow behaviors and to measure the flow asymmetry. The temperature imbalance is computed by taking the difference between the temperature at the center of the right gate exit T_2 and the temperature at the center of the left gate exit T_1 . A measure of the flow symmetry is then given by the time evolution of the temperature difference $\Delta T(t) = T_2(t) - T_1(t)$. A $\Delta T(t)$ that remains small enough when compared with the temperature scale ΔT_0 is interpreted as corresponding to a stable and symmetric flow. If the temperature difference increases rapidly toward a value close to the temperature scale, then the flow is said to be unstable. Finally, when ΔT becomes larger than $10^{-3}\Delta T_0$, but does not reach a plateau before the end of the simulation, we consider that the flow is in a transition state between a stable and an unstable behavior. The different results for ΔT and the classification of the flow behavior will be used to establish the

Figure 7. Example of a regular and symmetric mesh with 1257 nodes.

stability chart of the flow as a function of the flow parameters Gz and B . The dimensionless numbers Gz and B have the following physical interpretation:

- The Graetz number is the ratio of heat transfer by convection in the direction of the length scale R to heat transfer by conduction in the direction of the length scale H . A low value of Gz indicates that there is sufficient time for a substantial cooling of the fluid as it spends more time in the injector. Conversely, a greater value of Gz indicates a cooling time that is too short to observe a marked reduction of the fluid temperature.
- The B parameter represents the ratio of temperature changes ΔT_0 inside the flow to the temperature scale characterizing rheology changes ΔT_{rheol} . For example, a low value of ΔT_{rheol} with respect to ΔT_0 indicates that the viscosity changes induced by the temperature are important. Hence, B reflects the sensitivity of viscosity to temperature changes within the flow. For example, increasing the value of B has the effect of increasing the variation of the viscosity for a given difference in temperature inside the injector.

Note that using regular and symmetric meshes offer the advantage of having symmetric nodes, thus giving precise temperature measures even when temperature variations are small. The time evolution of the dimensionless temperature differences ΔT for various combinations of Gz and B values is shown in Figure 8. The total computational time is considered to be the filling time of a hypothetical mold cavity of volume equal to 25 times the gate volume. This gives $t_{\text{fill}} = 25t_{\text{ref}}$ where $t_{\text{ref}} = 4HR/Q$ and Q denotes the flow rate. For the present conditions we obtain $t_{\text{fill}} = 87.5$. The time step is set to $\Delta t = 0.175$, which leads to 500 time steps. As can be seen, stable, unstable and in transition flows can be found in Figure 8. For example, stable flow is given by the ΔT curve for $B = 2$ and $Gz = 0.5$, whereas an unstable flow is obtained for $B = 4$ and $Gz = 10$. When $B = 4$ and $Gz = 20$, the flow is in transition, i.e. the flow is not stable, but the asymmetry in temperature develops slowly over time and does not reach a steady state before the end of the simulation. Details of these three cases are discussed hereafter.

4.2.4. Stable flow: $B = 2$ and $Gz = 0.5$. Figure 9 shows the contour lines of the temperature and the u component of the velocity at the end of the simulation ($t = t_{\text{fill}}$). It is clearly seen that the

Figure 8. Time evolution of ΔT for various Gr numbers with: (a) $B=1$; (b) 2; (c) 3; and (d) 4.

Figure 9. Temperature and velocity contour lines for $B=2$ and $Gr=0.5$:
(a) temperature and (b) velocity u .

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Figure 10. Temperature and velocity contour lines for $B=4$ and $Gz=10$:
(a) temperature and (b) velocity u .

flow is stable as it remains symmetric for the entire simulation time. The low value of Gz (i.e. 0.5) implies that the fluid is cooled down before entering the injector gate. The flow speed is low and the conduction is high, driving the heat transfer toward the injector walls and resulting in high thermal gradients. However, the low value of B indicates that the temperature changes have little effect on the viscosity. Hence, the fluid viscosity is relatively uniform and thermal perturbations do not disturb the symmetry of the flow.

4.2.5. Unstable flow: $B=4$ and $Gz=10$. This example of unstable flow is shown in Figure 10. We can easily see that the initial symmetric flow is unstable. For this case the instability has reached a point where all the fluid exits the domain through the right branch. There is no flow in the other branch. The value of Gz is 20 times higher than for the previous stable flow, meaning that the principal mechanism for heat transfer is convection. At a higher value of B , the viscosity depends more on the temperature changes occurring in the flow (i.e. a higher sensitivity to it). The local cooling induced by temperature perturbations will increase the viscosity that will in turn decrease the fluid speed through one branch which in turn will allow further cooling on one side to the expense of the other side. This phenomenon feeds on itself until the flow ceases completely in one branch to flow out through the other.

4.2.6. Transition flow: $B=4$ and $Gz=20$. For these values of Gz and B , the flow is in transition between stable and unstable flow behavior as shown in Figure 11. Fluid exits through both sides yet exhibits asymmetry. Unlike the unstable flow, the transitional flow displays an asymmetry in temperature and speed that develops slowly over time and that does not reach a steady state before the end of the simulation. Here, the value of Gz is high, indicating a poor cooling of the fluid in the injector. The high speed of the flow leaves little time for the fluid to cool down before leaving the injector. Moreover, its potential to exhibit thermal instability is weakened. Hence, the

Figure 11. Temperature and velocity contour lines for $B=4$ and $Gz=20$:
(a) temperature and (b) velocity u .

flow does not have sufficient time to become significantly asymmetrical. Figure 8(d) shows that for higher values of Gz (i.e. 50 and more), the flow becomes stable again.

4.2.7. Flow stability chart. Based on the numerical results and the classification criterion described previously, we have established the stability chart of the flow as a function of Gz and B numbers (see Figure 12(a)). This chart also shows the boundary between stable and unstable flows. Each point on the stability chart is associated with one of the three possible flow behaviors. Filled squares identify stable flows, empty squares correspond to unstable flows, transient flows are marked by filled circles and finally the black curve is the limit of the region where the flow is unstable.

The stability chart defines three zones in terms of the Gz number:

- $0.1 \leq Gz \leq 0.25$;
- $0.25 < Gz < 20$;
- $20 \leq Gz \leq 200$.

In the first zone the flow is stable except for high values of B (i.e. greater than 8). Stability is improved by the fact that the flow is cooled down even before entering the gate. This may indicate that for very small values of Gz , the temperature scale ΔT_0 used to define B (Equation (23)) becomes inappropriate. To reflect the fast cooling of the fluid, we define a new temperature scale $\Delta T_0^* = T_a^* - T_r$, where T_a^* is measured where the flow splits in the horizontal part of the channel. With this new temperature scale, we can define a corrected dimensionless number $B^* = \Delta T_0^* / \Delta T_{\text{rheol}}$. Figure 12(b) shows the flow behavior as a function of flow parameters Gz and B^* . We observe that for a small value of Gz , B^* is much smaller than B . If B^* is small when Gz is low, then the sensitivity of the viscosity with respect to temperature is very low. For those conditions, the stability of the flow is preserved. Note that B^* and B have nearly the same value for higher values of Gz , because $T_a^* \sim T_a$.

In the second zone the flow becomes unstable for a wider range of values of the parameter B . In the instability pool large values of B indicate that thermally induced viscosity gradients are

Figure 12. (a) (Gz, B) stability chart and (b) (Gz, B^*) stability chart.

important. At the upper right corner of this region, we note the existence of a small zone where the flow is stable even when B is high. In this subzone the high flow speed has a stabilizing effect by carrying the instabilities out of the injector.

Finally, the third region of the stability chart shows that the flow is stable for all values of the parameter B if Gz is sufficiently large. For high values of Gz the heat transfer by convection is much more important than heat conduction. Hence, the flow is fast enough to prevent thermal perturbations from cascading into the flow and to cause instabilities. Using the definition of the Gz number (26), a family of gate geometry can be associated with the different values taken by this parameter. Indeed, Gz involves the ratio between the injector height H and its radius R (see Figure 2). For example, a small value of Gz implies a long and narrow horizontal channel for the injector (i.e. small H and high R). Based on the stability chart, we know the values of Gz for which the flow is stable or not. A family of gate geometry that gives stable flow can then be specified.

5. SENSITIVITY ANALYSIS

The boundary between regions of stable and unstable flows is defined by the flow stability chart obtained from the flow analysis. We now show how sensitivities computed from the SEM can correctly predict the flow response when the parameters Gz and B are perturbed around their nominal values. In other words, without any knowledge of the stability map, sensitivities provide information about whether a change in flow parameters results in the flow becoming more or less stable. Moreover, sensitivities can also indicate when the boundary between stable and unstable flows is being crossed. Hence, sensitivities can foresee the instability transition.

In this context, sensitivities are used to define the nature of a nearby solution. The first order approximation of nearby solutions of a function $F(a, b)$ is given by the following Taylor series expansion:

$$F(a + \Delta a, b + \Delta b) \approx F(a, b) + \frac{\partial F(a, b)}{\partial a} \Delta a + \frac{\partial F(a, b)}{\partial b} \Delta b \quad (49)$$

The response of F to a perturbation of a and b is given by $\partial F(a, b)/\partial a$ and $\partial F(a, b)/\partial b$. They express the sensitivity of F with respect to a and b . For example consider the sensitivity of ΔT with respect to parameters Gz and B . Recall that the flow behavior and its evolution are given by the ΔT curves. Replacing F by ΔT in Equation (49) leads to

$$\Delta T(t, Gz + \delta Gz, B + \delta B) \approx \Delta T(t, Gz, B) + s_{\Delta T}^{Gz} \delta Gz + s_{\Delta T}^B \delta B \quad (50)$$

where $s_{\Delta T}^{Gz} = \partial \Delta T / \partial Gz$ and $s_{\Delta T}^B = \partial \Delta T / \partial B$. The sensitivity $s_{\Delta T}$ is simply obtained by taking the difference between the temperature sensitivity computed at the exit points of the injector:

$$s_{\Delta T}^{Gz} = s_{T_2}^{Gz} - s_{T_1}^{Gz} \quad (51)$$

$$s_{\Delta T}^B = s_{T_2}^B - s_{T_1}^B \quad (52)$$

The temperature sensitivities with respect to Gz and B are obtained by solving the CSEs (34)–(36).

5.1. Verification by finite difference

Before going any further, we first verify the sensitivity calculation by comparing it with a finite difference approximation. The sensitivity of ΔT with respect to parameter a is given by

$$s_{\Delta T}^a = \frac{\partial \Delta T(\mathbf{x}, t; a)}{\partial a} \quad (53)$$

where a is either Gz , B or m . A second-order finite difference approximation of $s_{\Delta T}^a$ is given by

$$s_{\Delta T}^a \approx \frac{\Delta T(\mathbf{x}, t; a_0 + \delta a) - \Delta T(\mathbf{x}, t; a_0 - \delta a)}{2\delta a} \quad (54)$$

where δa is a small perturbation of a around its nominal value a_0 . The CSE sensitivities are compared with the finite difference approximation in Figure 13 for $B=2$ and $Gz=2$. As can be seen, the agreement is excellent. This indicates that the CSEs method yields accurate solutions and provides proper trends in the flow response to changes in parameter values.

5.2. Sensitivity curves and stability vector field

Figure 14 shows the sensitivity of ΔT with respect to Gz , m and B for $Gz=5$ and various values of B . The variation of ΔT with time is also shown as a reference. One way to interpret the sensitivity $s_{\Delta T}$ is to imagine the surface generated by the ΔT curves distributed along a third axis (i.e. B axis) for various values of B . If we move along the B axis for a fixed value of t/t_{fill} , then the slope is given by the sensitivity $s_{\Delta T}^B$.

To determine if the sensitivities are able to predict the change in the solution toward a more stable or less stable flow when flow parameters are changed we rewrite Equation (50) as

$$\Delta T(t, Gz + \delta Gz, B + \delta B) \approx \Delta T(t, Gz, B) + \nabla(\Delta T(t, Gz, B)) \cdot \delta \quad (55)$$

where $\nabla(\Delta T) = (s_{\Delta T}^{Gz}, s_{\Delta T}^B)$ and $\delta = (\delta Gz, \delta B)^T$. Hence, $s_{\Delta T}^{Gz}$ and $s_{\Delta T}^B$ may be seen as components of a vector, i.e. the gradient of ΔT in the Gz - B parameter space. We denote this vector of sensitivities with $\mathbf{v}_s$. It will also be referred latter in the paper as the stability vector because it can be used to indicate the direction towards increased instability. Equation (55) indicates that

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Figure 13. (a) Time evolution of ΔT for $B=2$ and $Gz=2$; (b) sensitivity of ΔT with respect to Gz ; (c) m ; and (d) B , obtained by CSE and computed by finite difference with $\delta a=0.001a$.

when the parameters Gz and B are modified such that the dot product between the vectors $\mathbf{v}_s$ and δ is positive, then the temperature imbalance increases (see Figure 15(a)) indicating that the flow becomes more unstable. Hence, we would expect that if one follows the direction from a more stable flow to a less stable flow (solid arrows on Figure 15(b)), then we should observe that $\Delta T(t, Gz + \delta Gz, B + \delta B) > \Delta T(t, Gz, B)$, corresponding to parameters changing in the direction of the vector of sensitivities. On the other hand, taking the opposite direction (i.e. dashed arrows) yields $\Delta T(t, Gz + \delta Gz, B + \delta B) < \Delta T(t, Gz, B)$, which means that the flow becomes more stable. We can think of the vector whose components are the sensitivities of the flow variables as a stability vector pointing towards regions of increasing instability.

Hereafter, we will use the normalized stability vectors $\bar{\mathbf{v}}_s$ defined as:

$$\bar{\mathbf{v}}_s = \left(\frac{s_{\Delta T}^{Gz}}{N}, \frac{s_{\Delta T}^B}{N} \right) \quad (56)$$

where N is given by

$$N = \|\mathbf{v}_s\| = \sqrt{(s_{\Delta T}^{Gz})^2 + (s_{\Delta T}^B)^2} \quad (57)$$

Figure 14. (a) Time evolution of ΔT for $Gz=5$; (b) sensitivity of ΔT with respect to Gz ; (c) m ; and (d) B , obtained by CSE for various values of B .

The amplitude of the stability vectors varies within a wide range of values and therefore using normalized values makes the charts more readable. The stability vectors are time dependent, thus making the interpretation of the stability behavior more difficult. Different possibilities to define the stability vectors are available to us: one can choose values defined at a specific time or can pick values that have a particular meaning. The first approach is not very useful, as flow instabilities evolve faster or slower over time depending on the choice of Gz and B . Several combinations were tested and it was found that the most representative is the one based on the maximum absolute values of the sensitivity curves. This choice is motivated by the fact that those values indicate the maximum flow variations.

Once all vector components are extracted from the sensitivity curves, we can cover the stability chart with a vector field that defines what we call the stability vector field (see Figure 16). Globally, vectors outside of the instability region are pointing towards the unstable zone as expected, while those located within the zone of instability generally indicate the region where the flow is most unstable. Moreover, the most unstable flows are found along the dashed line where the stability vectors are pointing toward each other. Stability vectors are therefore able to predict the flow behavior by providing the direction to follow to move from a point where the flow is more stable, to another point where the flow is less stable.

Figure 15. Interpretation of stability vectors: (a) relation of sensitivity vectors and stability and (b) stability chart.

Figure 16. Stability vector field as a function of flow parameters Gz and B .

5.3. Magnitude of stability vectors chart

The stability vector chart focuses on the orientation of the stability vectors v_s . Additional important information about the flow behavior can be obtained from the magnitude N of the stability vectors as given by Equation (57).

Figure 17. 3D chart representing the magnitude of the stability vectors: (a) complete set of data and (b) detail for $(0.5 \leq Gz \leq 2$ and $1 \leq B \leq 12)$.

Figure 17 presents a 3D plot of N as a function of parameters Gz and B . On the Gz - B plane, we have the familiar stability chart. Over each point of the stability chart, we can see a needle that represents the value of N . When the flow is stable (filled squares), N is small, whereas N takes large values when the flow is unstable or in transition (empty squares and filled circles). Furthermore, the highest values are found on or near the transition curve. To better see the changes of N , a limited section of the chart is shown on Figure 17(b). We clearly see that the highest values of N are near the boundary between stable and unstable flows. Indeed, if we look at Figure 14, then we see that sensitivities are maximum at transition points, or near these points. This indicates that the solution exhibits more important changes with respect to flow parameters in the transition zone. The sensitivities are able to detect this behavior and to circumscribe the unstable region without any direct reference to the ΔT curves.

6. CONCLUSION

This paper presented a study of the flow of a generalized Newtonian fluid inside a symmetrical geometry presenting both symmetric and non-symmetric solutions. Instabilities in the flow are shown to be caused by the dependence of the viscosity on the shear rate and temperature. A flow stability chart in terms of the dimensionless numbers Gz and B indicates two distinct regions corresponding to stable and respective unstable flow. The flow behavior can also be interpreted by computing flow sensitivities which are shown to present larger magnitudes in the vicinity of the transition between stable and unstable flows. As more than one parameter affects the flow, the resulting sensitivity vectors are shown to indicate the direction towards increased instability. Increased stability of the flow can then be obtained by modifying the flow parameters in the direction opposite to that of the sensitivity vectors. Sensitivities are therefore a powerful tool in flow control of flow problems that can be subject to thermo-hydrodynamic instabilities.

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