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EFFECT OF STIFFNESS TAPER IN ASEISMIC DESIGN

BY

G. N. BYCROFT

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EFFECT OF STIFFNESS TAPER IN ASEISMIC DESIGN

By G. N. BYCROFT

ABSTRACT

An investigation is made of the effect of changing the stiffness distribution up the height of a linear shear framed structure when subjected to idealized earthquake motions. The mean value of the largest strains arising in successive earthquakes is determined together with the associated probability distribution. It appears that the chances of finding a strain value greater than twice the mean are very small.

INTRODUCTION

FRAMED structures subjected to horizontal ground motions deform mainly by bending of the vertical columns producing horizontal translation of the floors and hence a shearing deformation of the structure. In a pinjointed frame the shear forces come from direct tension and compression of the cross-ties. The vibration of the structure is compounded of the individual vibrations of the modes of oscillation of the structure and is a function of the ground motion, the mass and stiffness distributions up the structure and the damping ratios of the modes.

This investigation is concerned with examining the effect of changing the shear stiffness up the height of the structure.

Various assumptions have been made. The mass distribution up the structure has been taken as constant. This is approximately true for a number of structures. The damping ratios of the modes of oscillation of the structure have been made equal to conform with what little is known about structural damping. In another paper¹ the author suggested that large earthquake ground accelerations be represented by finite bursts of "white noise". This is the type of representation considered here. Initially, for the purposes of a little analysis, the mass and stiffness distribution are taken as continuous up the structure and the stiffness variation assumed is linear with respect to height.

ANALYSIS

l = height of structure

x = distance from top of the structure to some point down its height.

y = horizontal displacement of this point from its equilibrium position relative to the base

λ = mass per unit height of the structure

k_0 = shear stiffness at the top

$k_0(1 + \rho x/l)$ = shear stiffness at a distance x from the top and is the assumed variation of stiffness.

Each value of the parameter ρ determines a particular family of structures.

y_0 = displacement of the ground in space.

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¹ "White noise representation of earthquakes", G. N. Bycroft, *Proc. A.S.C.E.*, 86 (EM-2): 1-16 (1960).

The equation of motion of this system follows simply as,

$$\frac{\partial}{\partial x} \left[(1 + \rho x/l) \frac{\partial y}{\partial x} \right] = \frac{\lambda}{gk_0} \left[\frac{\partial^2 y}{\partial t^2} + \frac{\partial^2 y_0}{\partial t^2} \right] \quad (1)$$

If we substitute $\lambda/gk_0 = \alpha^2$, $(1 + \rho x/l) = X$, $y + y_0 = Y$, then

$$\frac{\partial}{\partial X} \left(X \frac{\partial Y}{\partial X} \right) = \frac{\alpha^2 l^2}{\rho^2} \frac{\partial^2 Y}{\partial t^2} \quad (2)$$

A Laplace transformation of this expression with respect to time gives,

$$X \frac{\partial^2 \bar{Y}}{\partial X^2} + \frac{\partial \bar{Y}}{\partial X} - \frac{\alpha^2 l^2 p^2 \bar{Y}}{\rho^2} = 0 \quad (3)$$

where $\bar{Y}$ is the transform of Y in the transform variable p .

Substitute $z = \sqrt{X}$ and

$$\frac{\partial^2 \bar{Y}}{\partial z^2} + \frac{1}{z} \frac{\partial \bar{Y}}{\partial z} - \frac{4\alpha^2 l^2 p^2 \bar{Y}}{\rho^2} = 0 \quad (4)$$

This is the zero order Bessel equation and the solution is,

$$\bar{Y} = AJ_0 \left(\frac{2 \alpha l i p}{\rho} \sqrt{1 + \rho x/l} \right) + BY_0 \left(\frac{2 \alpha l i p}{\rho} \sqrt{1 + \rho x/l} \right) \quad (5)$$

The relevant boundary conditions are that when $x = l$, $y = 0$, i.e., $\bar{Y} = \bar{y}_0$. At the top the shear force is zero, i.e., when $x = 0$,

$$\frac{\partial y}{\partial x} = \frac{\partial Y}{\partial x} = \frac{\partial \bar{Y}}{\partial x} = 0$$

The transformed displacement is now

$$\bar{Y} = \frac{\bar{y}_0 \left[J_1 \left(\frac{2 \alpha l i p}{\rho} \right) Y_0 \left(\frac{2 \alpha l i p}{\rho} \sqrt{1 + \rho x/l} \right) - Y_1 \left(\frac{2 \alpha l i p}{\rho} \right) J_0 \left(\frac{2 \alpha l i p}{\rho} \sqrt{1 + \rho x/l} \right) \right]}{p^2 \left[J_1 \left(\frac{2 \alpha l i p}{\rho} \right) Y_0 \left(\frac{2 \alpha l i p}{\rho} \sqrt{1 + \rho} \right) - Y_1 \left(\frac{2 \alpha l i p}{\rho} \right) J_0 \left(\frac{2 \alpha l i p}{\rho} \sqrt{1 + \rho} \right) \right]} \quad (6)$$

where $\bar{y}_0$ has been replaced by $\bar{y}_0/p^2$, i.e. the transform of the base acceleration.

The roots of the denominator of this expression give the eigenvalues of the system. These roots, p_n , are imaginary and ip_n is real. If p_n is a root, then so is $-p_n$. This may be shown by using

$$J_p(z i^{2m}) = i^{2mp} J_p(z) \quad (7)$$

and

$$Y_p(zi^{2m}) = i^{-2mp}Y_p(z) + 2i \sin(mp\pi) \operatorname{ctg}(p\pi)J_n(z) \quad (8)$$

and taking the appropriate values of p , m , and the limits.

Equation (6) is inverted by a combination of the inversion and convolution theorems.

The residue from the double pole $p = 0$ is zero. The residues from the poles $\pm ip_n$ are found by differentiation of the denominator and introduction of the poles. The function really wanted is not the displacement Y but the strain,

$$\frac{\partial y}{\partial x} = \frac{\partial Y}{\partial x}.$$

Fig. 1. Variation of Fundamental Frequency Factor X_1 , with Parameter ρ .

After the necessary reduction this follows, as,

$$\begin{aligned} & \rho[J_1(X_n)Y_1(X_n\sqrt{1+\rho x/l}) \\ & - Y_1(X_n)J_1(X_n\sqrt{1+\rho x/l})] \int_0^t \ddot{y}_0(\tau) \sin p_n'(t-\tau) d\tau \quad (9) \\ \frac{\partial y}{\partial x} = & \sum_{n=1}^{\infty} \frac{(1+\rho x/l)^{1/2} l p_n' [J_0(X_n)Y_0(X_n\sqrt{1+\rho}) - Y_0(X_n)J_0(X_n\sqrt{1+\rho}) \\ & - \sqrt{1+\rho} \{J_1(X_n)Y_1(X_n\sqrt{1+\rho}) - Y_1(X_n)J_1(X_n\sqrt{1+\rho})\}]} \end{aligned}$$

where $p_n' = ip_n$ and $X_n = 2alp_n'/\rho$ and is a function solely of ρ . The relation of X_1 and ρ is shown in fig. 1. When X_n is large, as for the higher harmonics, the Bessel functions in the characteristic equation may be replaced by their asymptotic values and then,

$$X_n = \frac{(n - \frac{1}{2})\pi}{(\sqrt{1+\rho} - 1)}$$

Equation (9) is the summation of the responses of the various modes of the system. The ground motion occurs only in the factor

$$\frac{1}{p_n'} \int_0^t \ddot{y}_0(\tau) \sin p_n'(t - \tau) d\tau$$

and the maximum value of this expression is the "displacement spectrum" of the earthquake. If the n th mode of vibration is damped by a factor q_n of critical damping then the above undamped spectrum is replaced by the damped one,

$$\frac{1}{p_n'} \int_0^t \ddot{y}_0(\tau) e^{-q_n p_n' (t - \tau)} \sin p_n'(t - \tau) d\tau$$

The coefficients in front of the integrals give the "participation factors" of the modes. Equation (9) could be used to calculate the strains arising for any particular earthquake acceleration $\ddot{y}_0(t)$.

This would be extremely tedious and not especially profitable unless carried out for a large number of earthquakes. The alternative, discussed in reference 1, is to regard earthquakes as bursts of random accelerations of constant spectral density, i.e. "white". A suitable measure of the relative size of the vibration up the structure is the r.m.s. value and this may be obtained from known properties of random vibrations. The following analysis determines the r.m.s. value of the strain when $\ddot{y}_0(t)$ is a continuous white noise acceleration of spectral density S_0 .

If $\ddot{y}_0(t)$ is replaced by $e^{i\omega t}$ in equation (9) the complex frequency response of the strain follows as

$$H(w) = \sum_{n=1}^{\infty} \frac{\rho A_n}{(1 + \rho x/l)^{1/2} l p_n' \left[\left(1 - \frac{w^2}{p_n'^2} \right) + i 2q \frac{w}{p_n'} \right]} \quad (10)$$

where

$$A_n = \frac{[J_1(X_n) Y_1(X_n \sqrt{1 + \rho x/l}) - Y_1(X_n) J_1(X_n \sqrt{1 + \rho x/l})]}{[J_0(X_n) Y_0(X_n \sqrt{1 + \rho}) - Y_0(X_n) J_0(X_n \sqrt{1 + \rho}) - \sqrt{1 + \rho} \{ J_1(X_n) Y_1(X_n \sqrt{1 + \rho}) - Y_1(X_n) J_1(X_n \sqrt{1 + \rho}) \}]} \quad (11)$$

The root mean square strain is given by

$$\left\{ \overline{\left(\frac{\partial y}{\partial x} \right)^2} \right\}^{1/2} = \left[\frac{S_0}{2\pi} \int_0^{\infty} |H(w)|^2 dw \right]^{1/2} \quad (12)$$

If the common mode damping ratio q is not large and the frequencies of the modes are reasonably separated, as is the case here, the cross terms arising in $|H(w)|^2$ may be neglected. A contour integration then determines the root mean square strain as,

$$\left\{ \overline{\left(\frac{\partial y}{\partial x} \right)^2} \right\}^{1/2} = \left[\frac{S_0}{8q} \sum_{n=1}^{\infty} \frac{\rho^2 A_n^2}{(1 + \rho x/l) l^2 p_n'^3} \right]^{1/2} \quad (13)$$

If we wish to compare the r.m.s. strains in structures having the same distributed mass and the same base stiffness but different stiffness taper ρ then equation (13) may be written as

$$\left\{ \left(\frac{\partial y}{\partial x} \right)^2 \right\}^{1/2} = \frac{S_0^{1/2} l^{1/2} \lambda^{3/4}}{q^{1/2} g^{3/4} K^{3/4}} S_K(\rho, x/l) \quad (14)$$

where $K = k_0(1 + \rho)$ and is the common base stiffness, and

$$S_K = \left[\sum_{n=1}^{\infty} \frac{A_n^2 (1 + \rho)^{3/2}}{(1 + \rho x/l) X_n^3 \rho} \right]^{1/2} \quad (15)$$

Fig. 2. Strain Factor $S_K(\rho, x/l)$.

If we wish to compare structures having the same mass distribution and same fundamental period T_1 then equation (13) is written as

$$\left\{ \left(\frac{\partial y}{\partial x} \right)^2 \right\}^{1/2} = \frac{S_0^{1/2} T_1^{3/2}}{q^{1/2} 8\pi^{3/2} l} S_T(\rho, x/l) \quad (16)$$

where

$$S_T(\rho, x/l) = \frac{\rho^{3/2} X_1^{3/2}}{(1 + \rho)^{3/4}} S_K(\rho, x/l) \quad (17)$$

These two functions have been calculated and are shown in figures 2 and 3. As would be expected a uniform structure, given by $\rho = 0$, suffers the greatest strain

at the base. As the stiffness is tapered off, more strain is thrown on to the upper regions of the structure. For a completely tapered structure given by $\rho = \infty$ a whipping effect occurs at the top and the strain here becomes unbounded. Fig. 2 shows that structures having the same mass and base stiffness experience approximately the same base strain no matter how the stiffness tapers off. Further, the variation of strain up the structure is not critically dependent on the parameter ρ except towards the top. The value $\rho = 3.0$ gives approximately equal strain over most of the height.

Fig. 3. Strain Factor $S_T(\rho, x/l)$.

Of interest is the variation of the r.m.s. shear force up the structure. For structures with a common base stiffness this is given by

$$(1 + \rho x/l) k_0 \left\{ \left(\frac{\partial y}{\partial x} \right)^2 \right\}^{1/2} = \frac{S_0^{1/2} l^{1/2} \lambda^{3/4} K^{1/4} (1 + \rho x/l) S_K(\rho, x/l)}{q^{1/2} g^{3/4} (1 + \rho)} \quad (18)$$

The non-dimensional r.m.s. shear force may then be defined as

$$F_K = \frac{(1 + \rho x/l)}{(1 + \rho)} S_K(\rho, x/l) \quad (19)$$

This factor is shown in fig. 4 and it is seen that changing the stiffness taper up the structure over the full range of $\rho = 0$ to $\rho = \infty$ does not have a great effect on the shear forces arising.

A suitable design will depend on what permissible strains are allowed in the structure and on the relation of strength of the structure to the shear forces arising. The former criterion depends largely on the permissible deformation that the structure's cladding will stand without damage and will largely be a constant value up the height of the structure. The latter criterion depends on the relation of the

stiffness at any height to the strength there. This depends on the type of construction. In a pin-jointed structure with the same geometrical configuration of ties up its height the strength will be proportional to the stiffness. If the strength is denoted by D and the base strength by D_b then,

$$D = D_b \frac{(1 + \rho x/l)}{(1 + \rho)} \quad (20)$$

This has been plotted as the dashed lines in fig. 4 where the base strength has been made equal to the base shear force arising. In this situation a uniform structure $\rho = 0$ is very much overdesigned and a completely tapered structure $\rho = \infty$ is

Fig. 4. Shear Force Factor $F_K(\rho, x/l)$ and Strength $D(\rho, x/l)$.

underdesigned. When $\rho = 3$, the shear forces and the strength are fairly closely the same except in the top 10% of the height. As $\rho = 3$ also makes the strain largely constant up the height, this may be regarded as a suitable design for these circumstances.

The probability distribution of instantaneous values of strain and shear force is Gaussian and has the associated well-known properties.

Although the r.m.s. values of stress and strain arising when the structure is subjected to a continuous white noise excitation is a useful means of comparing the relative sizes of the vibration at different points of the structure, it does not provide all the information required.

In a linearly elastic system failure will depend on exceeding some particular value of stress or strain.

Earthquakes are of finite duration and what is really wanted is a measure of the largest stress or strain that arises during successive bursts of the white noise excitation corresponding to successive earthquakes together with the associated probability distribution. To the best of the author's knowledge, there is no analysis

available which determines these features. In order to examine this aspect of the problem the situation was simulated on an analog computer.

MEAN VALUE AND PROBABILITY DISTRIBUTION OF LARGEST VALUES

Single Degree of Freedom.—Consider first the single degree of freedom case. The integral $\int_0^t W_m(\tau)e^{-qp(t-\tau)} \cos p(t-\tau) d\tau$ gives the instantaneous velocity and the integral $(1/p)\int_0^t W_m(\tau)e^{-qp(t-\tau)} \sin p(t-\tau) d\tau$ the instantaneous displacement of the damped oscillator when subjected to an acceleration $W_m(\tau)$. The largest values of these integrals are known as the damped velocity and displacement spectra of the particular excitation $W_m(\tau)$. If $W_m(\tau)$ is a series of random acceleration excitations of the same duration and same spectral density ("white noise") but, of course,

Fig. 5. Velocity Spectra, $S_0 = 0.75 \text{ ft}^2/\text{sec}^4/\text{cps}$ and of 25 seconds Duration.

different configurations, the largest values of velocity and displacement occurring during the passage of the particular excitation will all be different. It is the mean value and the probability distribution of these largest values that we are interested in. A simple damped oscillator was simulated on an analog computer and fed with a white noise source.

The mean largest velocity $\overline{\partial y_m/\partial t}$ occurring during each of a number of bursts of excitation was determined and is shown in fig. 5 as a function of period for three values of damping ratio q . It is the close similarity of these damped spectra to Housner's averaged earthquake spectra² that is the justification for this representation. The particular size shown corresponds to the mean value of the spectrum of the earthquake at El Centro 1940 and this is given by an acceleration spectral density of $0.75 \text{ ft}^2/\text{sec}^4/\text{c.p.s.}$ and a duration of 25 seconds.

It is to be expected that the probability distributions of the largest displacement y_m or velocity $\partial y_m/\partial t$ are functions of period and damping. This is indeed the case

² "Limit Design of Structures to Withstand Earthquakes" G. W. Housner, Proc. 1st World Conf. on Earthquake Engineering, San Francisco, June 1956.

Fig. 6. Probability Distribution, $T = 0.20$. Fig. 7. Probability Distribution, $T = 3.0$.

but the effect of changing these parameters over ranges relevant in this investigation is found to be small. This is shown in figs. 6 and 7 for 5%, 10%, 20% damping and for two periods $T = 0.2$ and $T = 3.0$. The probability curves for periods between these values were found to lie between the two probability sets shown. These curves were constructed from 100 runs and show the probability $P(y_m > r \bar{y}_m)$ i.e. of the largest displacement or velocity in any one burst being r times greater than the mean largest. It is seen that the effect of changing damping and period over the relevant range shown is small and also the chances of even finding a value twice the mean is very small. Zero damping, however, shows a much greater dispersion of maxima.

Multi-degree of Freedom Case.—This investigation was extended to the multi-degree of freedom case. A lumped system of five stories was simulated on the computer. The masses of the floors were made equal and the variation with height of the stiffness between floors made linear, i.e. if k_n is the shear stiffness between the n th and $(n - 1)$ th floors then,

$$k_n = (1 + \rho x/l)k_0$$

where x is measured from the top of the structure to the midpoint between the n th and $(n - 1)$ th floors.

In a previous paper³ the author shows that by introducing viscous damping both in the relative motion of one floor to the next and in the relative motion of the floor to the base it is possible to approximately realize equal damping of the modes. This is used in the following simulation. Let

$$\begin{aligned} w_n &= \text{angular frequency of } n\text{th mass referred to stiffness } k_n, \\ y_n &= \text{displacement of } n\text{th mass relative to the base,} \\ R &= \text{damping ratio relative to the mass below,} \\ r &= \text{damping ratio relative to base,} \end{aligned} \quad (22)$$

then, the equation of motion of the n th mass is,

$$\ddot{y}_n + w_n^2(y_n - y_{n-1}) - w_{n+1}^2(y_{n+1} - y_n) + 2w_nR(\dot{y}_n - \dot{y}_{n-1}) - 2w_{n+1}R(\dot{y}_{n+1} - \dot{y}_n) + 2w_nr\dot{y}_n = W_m(t) \quad (23)$$

This may be scaled to give

$$\begin{aligned} \ddot{Y}_n &= \frac{100 W_m(t)}{w_n^2 S} + \frac{w_{n+1}^2}{w_n^2} [Y_{n+1} - Y_n] - (Y_n - Y_{n-1}) + 2R \frac{w_{n+1}}{w_n^2} \dot{Y}_{n+1} \\ &\quad - \left(2R \frac{w_{n+1}}{w_n} + 2R + 2r \right) \dot{Y}_n + 2R \frac{w_{n-1}}{w_n} \dot{Y}_{n-1}. \end{aligned} \quad (24)$$

Figure 8 shows the computer circuit for the n th cell satisfying equation (24). As in the single degree of freedom case, bursts of acceleration $W_m(t)$ of equal spectral density and of 25 seconds duration were fed into this circuit for various values of ρ , T and q and the mean largest relative displacement between stories $(y_{m,n} - y_{m,n-1})$ determined. Any particular value of ρ determines a family of structures having particular mode shapes and ratios of mode frequencies which are independent of T and q . It would seem likely that the variation of $(y_{m,n} - y_{m,n-1})$ would have a similar variation with ρ as shown in figure 3 and its variation with T and q would be similar to the single degree of freedom case shown in figure 5. In order to show that this is largely true the results are presented as the ratio of the mean largest relative displacement to the mean largest displacement of a single

³ "Yield Displacements in Multistory Aseismic Design", G. N. Bycroft, *Bull. Seism. Soc. Am.*, 50: 441-453 (1960).

 Fig. 8. Analog Computer Diagram of N th Story.

 Fig. 9. Ratio of Multi-story to Single-story Strain, $\rho = 0$.

oscillator of the same period and damping as the fundamental period of the multi degree of freedom case. The mean largest displacement $\bar{y}_m$ of the single oscillator is closely given by $(T/2\pi)[\partial \bar{y}_m / \partial t]$ from figure 5. The ratio $(y_{m,n} - y_{m,n-1}) / \bar{y}_m$ is shown in figures 9, 10, 11 for various periods, damping and values of ρ . On the same graph is shown the corresponding ratio of r.m.s. values. The r.m.s. displacement of a single oscillator is given by,

$$(\bar{y}_m^2)^{1/2} = \frac{S_0^{1/2} T_1^{3/2}}{q^{1/2} 8\pi^{3/2}} \quad (25)$$

Fig. 10. Ratio of Multi-story to Single-story Strain, $\rho = 3.0$. Fig. 11. Ratio of Multi-story to Single-story Strain, $\rho = 10$.

and consequently it is seen from equation (16) that the required r.m.s. ratio for a five storied structure is given by

$$\frac{l}{5} \left[\left(\frac{\partial y}{\partial x} \right)^2 \right]^{1/2} \frac{1}{\{y_m^2\}^{1/2}} = \frac{S_T(\rho, x/l)}{5} \quad (26)$$

It is seen from the graphs that these ratios are similar for the range of parameters relevant to the problem. It should not be inferred, however, that this is a law

applicable to all damping ratios and ratios of duration of the excitation to period of the structure. By using the results of figures 3 and 5, it is thus possible to obtain the average relevant displacements and strains in a structure of this type. Figure 5, of course, refers to a particular size and duration of earthquake but may be immedi-

Fig. 12. Multi-story Probability Distribution, $T = 3.0$, $\rho = 3$, $q = 0.10$.

Fig. 13. Multi-story Probability Distribution, $T = 1.0$, $\rho = 0$, $q = 0.10$.

ately scaled to any other size. Figures 9, 10, 11 refer to a five storied structure but may be interpolated for other number of stories.

As in the single degree of freedom case, the associated probability distribution of largest values was determined and is shown in figures 12, 13, 14, 15, 16 for various parameters.

Only the top and bottom floors are shown, the rest having similar distributions. These distributions are similar to each other and to the single degree of freedom case in that the dispersion is small and the chances of finding an amplitude greater

Fig. 14. Multi-story Probability Distribution, $T = 0.5$, $\rho = 0$, $q = 0.05$. Fig. 15. Multi-story Probability Distribution, $T = 0.5$, $\rho = 3.0$, $q = 0.10$. Fig. 16. Multi-story Probability Distribution, $T = 0.5$, $\rho = 10$, $q = 0.10$.

than $1\frac{1}{2}$ times the average is small. The chances of finding an amplitude twice the average appear negligible.

DESIGN $\bar{g}$ VALUES

Finally, the mean largest equivalent static design acceleration value may be estimated. Referred to the mass above the section under consideration and expressed as a fraction of gravity, this may be shown to be

$$\bar{g} = \frac{8\pi(1 + \rho x/l)lS_T(\rho, x/l)}{xgX_1^2\rho^2 T_1} \left(\frac{\partial y_m}{\partial t} \right) \quad (27)$$

Figure 17 shows $\bar{g}$ for two different cases.

Fig. 17. Mean Equivalent Static Acceleration, $\bar{g}$.

These are mean largest values and should be increased by a factor of 1.5 to 2.0 to allow for the probability distribution. As is usually found in a linear analysis, these values are bigger than those specified by building codes.

The fact that the smaller values used in practice are satisfactory is due to several causes. The size of the earthquake represented by figure 5 is a large one, and will occur infrequently. Safety factors introduced are also part of the explanation. But probably the greatest safety factor comes from plastic yielding of the materials. Most materials may safely yield a few times every fifty or more years without failure. The large hysteretic dissipation of energy involved is probably the main factor which keeps the vibrations within safe limits.

CONCLUSIONS

(1) Tapering off the stiffness up the height of a structure does not have a great effect on the shear forces arising. It has considerable effect, however, on the resulting strains. A suitable design will depend on the relation of the strength to the stiffness of the structure and on the permissible strains which can be tolerated.

(2) For the particular type of structure considered the variation of the mean

largest stress and strain with period and damping is approximately the same in the multi-degree of freedom case as in the single degree of freedom case. Also the variation of mean largest stress and strain with parameter ρ is similar to the variation of r.m.s. stress and strain. Thus by use of Figures 3 and 5 mean design strains and stresses may be evaluated. The probability of finding a value twice the mean largest is very small.

(3) It appears that this general method should apply equally well to other structural systems having well separated mode frequencies.

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