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Brittle to ductile transition during indentation of ice

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The brittle to ductile transition in the deformation behaviour of ice is considered for indentation by a flat, rigid indenter under fully elastic plane strain or plane stress conditions. It is assumed that failure occurs when a crack initiated near an edge of the indenter is propagated immediately by the maximum shear stress at the site. Consideration is given to the relaxation of the elastic moduli and the change in the critical stress intensity factor for crack propagation that occur in the transition range of strain rates. The analysis shows that for the assumed conditions, the indentation pressure varies inversely with the square root of the width of the indenter, and increases to a maximum with decreasing strain rate. It also indicates that the necessary and sufficient conditions for brittle failure of ice are a critical combination of shear strain and strain rate that will cause a crack to form, which propagates immediately under the imposed stress.

Key words: ice, elasticity, indentation, brittle to ductile transition, fracture mechanics, failure.

La transition de fragile à ductile dans le comportement en déformation de la glace est étudiée dans un cas de pénétration à l'aide d'un pénétrateur rigide plat, en présence de déformations élastiques planes ou de contraintes planes. On croit qu'il y a rupture lorsqu'une fissure qui a pris naissance à proximité de la bordure du pénétrateur, se propage immédiatement, en raison de la contrainte de cisaillement maximale in situ. La relaxation du module d'élasticité et la modification du coefficient d'intensité de la contrainte critique pour la propagation des fissures ont été étudiées. L'analyse démontre que la pression de pénétration varie en raison inverse de la racine carrée de la largeur du pénétrateur, et augmente à son maximum lorsque la vitesse de déformation décroît, compte tenu des conditions hypothétiques. Elle indique également que les conditions nécessaires et propices à la rupture de la glace sont une combinaison critique de déformation tangentielle et de vitesse de déformation qui entraîneront la formation d'une fissure qui se propagera immédiatement en raison de la contrainte imposée.

Mots clés : glace, élasticité, pénétration, transition de fragile à ductile, mécanique de la rupture, rupture.

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Introduction

The brittle to ductile transition is an important feature of the mechanical behaviour of ice. It depends on several factors, which include strain rate, stress state, temperature, and type of ice. For simple compression at -10°C , it occurs over the strain rate range of about 10^{-2} to 10^{-3} s^{-1} . Strain rates of this order are common for many situations for which forces imposed by ice must be considered.

The immediate response of ice to an applied stress is elastic. This is followed by time-dependent delayed elastic and permanent deformation (e.g., Sinha 1978). It is reasonable, therefore, to consider the transition in deformation behaviour to be from brittle to ductile. Looking at the problem from this point of view may have the advantage that elastic theory and well-defined elastic properties can be used to investigate the deformation of ice prior to and into the transition region.

The relatively simple problem of a rigid, flat indenter pressed into a semi-infinite ice body will be used to identify and discuss features of the transition. It will be shown that the behaviour predicted by elastic theory, in particular the dependence of the load at failure on the width of the indenter, is consistent with field and laboratory measurements.

Theory

Muskhelishvili (1953) has shown that for conditions of plane strain or plane stress, the stress induced by a rigid, flat indenter of width $2l$, when pressed into the plane surface of an elastic half-space, can be determined from the stress function

$$[1] \quad \phi(Z) = \frac{iP_0}{2\pi} (Z + l)^{-1/2 + i\beta} (Z - l)^{-1/2 - i\beta}$$

where P_0 is the load per unit length on the indenter; $Z = x + iy$ is the position in the complex plane at which the stress is to be determined; and

$$[2] \quad \beta = \frac{\ln \chi}{2\pi}$$

in which

$$[3] \quad \chi = 3 - 4\nu \quad \text{for plane strain condition}$$

$$[4] \quad \chi = \frac{3 - \nu}{1 + \nu} \quad \text{for plane stress condition}$$

and ν is the Poisson's ratio. The assumption of a plane strain condition would be appropriate for an indenter of length much greater than its width, and that of plane stress for a relatively wide indenter pressed into the edge of an ice plate whose width is appreciably greater than its thickness.

Consider the physical arrangement shown in Fig. 1. The stresses at Z determined from [1] and expressed in non-dimensional form are

$$[5] \quad S_{xx} \left(\frac{2\pi l}{P_0} \right) = - \frac{1}{\sqrt{\alpha_1 \alpha_2}} \times [3 e^{\beta(\theta_2 - \theta_1)} \sin Z_1 + e^{-\beta(\theta_2 - \theta_1)} \sin Z_2] + \frac{2\alpha_1 \sin \theta_1 e^{\beta(\theta_2 - \theta_1)}}{(\alpha_1 \alpha_2)^{3/2}} [A \cos Z_3 - B \sin Z_3]$$

NOTE: Written discussion of this paper is welcomed and will be received by the Editor until August 31, 1991 (address inside front cover).

FIG. 1. Rigid indenter of width $2l$, pressed into the surface of an elastic half-space by a load P_0 per unit length.

$$[6] \quad S_{yy} \left(\frac{2\pi l}{P_0} \right) = \frac{1}{\sqrt{\alpha_1 \alpha_2}} [e^{-\beta(\theta_2 - \theta_1)} \sin Z_2 - e^{\beta(\theta_2 - \theta_1)} \sin Z_1] \\ - \frac{2\alpha_1 \sin \theta_1 e^{\beta(\theta_2 - \theta_1)}}{(\alpha_1 \alpha_2)^{3/2}} [A \cos Z_3 - B \sin Z_3]$$

$$[7] \quad S_{xy} \left(\frac{2\pi l}{P_0} \right) = \frac{1}{\sqrt{\alpha_1 \alpha_2}} [e^{-\beta(\theta_2 - \theta_1)} \cos Z_2 - e^{\beta(\theta_2 - \theta_1)} \cos Z_1] \\ - \frac{2\alpha_1 \sin \theta_1 e^{\beta(\theta_2 - \theta_1)}}{(\alpha_1 \alpha_2)^{3/2}} [A \sin Z_3 - B \cos Z_3]$$

where

$$\alpha_i = r_i/l$$

$$Z_1 = \beta \ln(r_2/r_1) + (\theta_2 + \theta_1)/2$$

$$Z_2 = \beta \ln(r_2/r_1) - (\theta_2 + \theta_1)/2$$

$$Z_3 = \beta \ln(r_2/r_1) + 3(\theta_2 + \theta_1)/2$$

$$A = 1 + \alpha_1 \cos \theta_1$$

$$B = \alpha_1 \sin \theta_1 + 2\beta$$

and the remaining terms are defined in Fig. 1. The major principal stress, σ_1 , at Z ; minor principal stress, σ_2 ; and the maximum shear stress, τ_m , are readily determined from S_{xx} , S_{yy} , and S_{xy} (Timoshenko and Goodier 1970).

Equations [5]–[7], and therefore those for the principal stresses, have a singularity at $x = \pm l$ and $y = 0$. The stresses S_{xx} , S_{yy} , and S_{xy} at the interface between the solid and the indenter near $x = l$ ($\theta_1 = \pi$) are plotted in Fig. 2. For $|x| > l$ and $y = 0$, $S_{yy} = S_{xy} = 0$.

Brittle failure

Cracks are observed to form in ice during the first 10^{-3} strain in unconfined compression when the applied stress exceeds about 0.5 MPa. They are long and narrow in columnar-grained ice and “penny shaped” in granular ice (Gold 1972a; Cole 1986). The average diameter of penny-shaped cracks and the width of cracks in columnar-grained ice are of the same order as the grain size (Cole 1986, 1989; Gold 1972b). Similar cracks are observed to form during indenta-

FIG. 2. Normalized stresses near edge $(l, 0)$ at the interface between the indenter and the elastic half-space, $\theta_1 = \pi$, $\nu = 0.3$.

tion; the first ones occurring near the edge of the indenter (Nuttall and Gold 1966; Frederking and Gold 1975; Michel and Toussaint 1977).

Deformation-induced crack initiation in polycrystalline ice subject to a compressive stress is considered to be caused by internal stress concentrations that occur in grain boundary regions. These stress concentrations develop because of the inability of grains to conform with the strain induced in the neighbouring grains (Gold 1963, 1972a; Hondoh and Higashi 1983). For a uniaxial compressive stress, cracks are observed to form when the stress and the strain exceed relatively critical values, which depend on the temperature and the strain rate (Kalifa *et al.* 1990). Their initiation is suppressed by a compressive minor principal stress (Sammonds and Murrell 1989). It would be expected, therefore, that the first crack to form under an indenter being pressed into an ice surface would be near an edge in the region where the major principal stress is large and the minor principal stress is relatively small. The location, size, and orientation of the crack would be determined primarily by the size and the relative orientation of the grains in the immediate vicinity of the edge, i.e., the crack will form at the first site that attains the critical conditions of stress and strain as the load is increased on the indenter. The position of this site would depend on the indenter width only to the extent that the width affects the size of the region in which the minor principal stress is less than a given value of the major principal stress.

Consider an indenter of width $2l$ indenting an ice surface at a rate sufficiently high to cause only elastic behaviour. For this condition, Poisson's ratio can be taken to be 0.30. It is assumed that (i) failure occurs when a crack is formed of suffi-

cient size to be propagated by the shear stress at the site of formation; (ii) the size of the crack and its distance from the edge of the indenter depend primarily on the grain size; (iii) the distance of the crack from the edge of the indenter is independent of the width of the indenter; and (iv) the crack forms in the region for which $\sigma_2/\sigma_1 < 0.1$.

The line defined by $\sigma_2/\sigma_1 = 0.1$ and Poisson's ratio of 0.30 is presented for the plane strain condition in Fig. 3 for edge 1, on a plot of α_1 vs. θ_1 . Shown also is the band $\theta_m \pm 5^\circ$, where θ_m is the angular coordinate for the maximum shear stress and given α_1 . It can be seen that for the fully elastic condition and $\alpha_1 < 0.1$ and $\theta_1 < 70^\circ$, the maximum shear stress is associated with a minor principal stress (compressive) that is less than $0.1\sigma_1$, i.e., the stress condition in the region where the shear stress is largest approaches that of an unconfined compression test. The situation for edge 2 is the mirror image of that for edge 1.

Assume a crack of a size large enough to propagate under the imposed maximum shear stress, τ_m , forms at a distance r_c from the edge of the indenter.

$$[8] \quad \tau_m = \frac{P_0}{2\pi l} F_r(\alpha_c, \theta_m, 0.30)$$

where $\alpha_c = r_c/l$ and $F_r(\alpha_c, \theta_m, 0.30)$ is determined from [5]–[7] using the appropriate values of α_c and θ_m and a Poisson's ratio of 0.30.

Consider two indentors, one of width $2l_1$ and the other of width $2l_2$, where the subscripts 1 and 2 denote the two situations. For two identical cracks forming at a distance r_c from an edge of each indenter, $\tau_{m1} = \tau_{m2}$ at failure. Therefore,

$$[9] \quad \frac{P_{01}}{2\pi l_1} F_r(\alpha_{c1}, \theta_{m1}, 0.30) = \frac{P_{02}}{2\pi l_2} F_r(\alpha_{c2}, \theta_{m2}, 0.30)$$

From [9], the ratio of the two imposed loads at failure is

$$[10] \quad \frac{P_{02}}{P_{01}} = \frac{l_2 F_{r1}}{l_1 F_{r2}}$$

The indentation pressure is given by

$$[11] \quad S_0 = \frac{P_0}{2l}$$

Combining [10] and [11] gives

$$[12] \quad S_{02} = S_{01} \frac{F_{r1}}{F_{r2}}$$

Assume that a load $P_{01} = 1 \text{ MN} \cdot \text{m}^{-1}$ is required to fail the ice under an indenter of width $2l = 1 \text{ m}$ when the initial crack forms at $r_c = 10^{-2} \text{ m}$. The indentation pressure at the interface between the indenter and the ice for this load is $S_{01} = 1 \text{ MPa}$. Corresponding values of the indentation pressure that would cause the crack to propagate when it is initiated in the same position under indentors of width 0.1, 10, and 100 m were calculated using [12] and are shown in Fig. 4. The same sets of calculations were made for the cases where the distance of the critical crack from the edge, r_c , is equal to 2×10^{-2} and $4 \times 10^{-2} \text{ m}$, and the results are presented also in Fig. 4. These calculations show that for the assumed criteria of failure, S_0/S_{01} has a marked dependence on the width of the indenter. In addition, they show that the slope of the dependence of $\ln(S_0/S_{01})$ on $\ln(2l)$ is relatively independent of the distance of the fracture initiating crack from the edge of

FIG. 3. Lines defined by $\sigma_2/\sigma_1 = 0.1$ for Poisson's ratios of 0.3 and 0.5, on a plot of normalized distance, α , and angle from edge 1. Shown also is the band given by $\theta_m \pm 5^\circ$ for both values of Poisson's ratio, where θ_m is the angular position of the maximum shear stress.

FIG. 4. Normalized indentation pressure to cause failure under an indenter of width $2l$, due to a crack formed at the given position r , and propagated by a specified maximum shear stress given by r_c ; $\nu = 0.3$.

the indenter assumed for each set of calculations and, therefore, of the value of the critical shear stress.

It would be expected that the position of the initiation of a crack causing failure would be relatively random within a range of distance from an edge. The calculations were repeated with the assumption that the critical shear stress for propagation was that given by the condition $r_c = 10^{-2} \text{ m}$, $2l = 1 \text{ m}$, and $P_0 = 1 \text{ MN} \cdot \text{m}^{-1}$, but that the crack causing failure was formed 2×10^{-2} or $4 \times 10^{-2} \text{ m}$ from the edge. The results, plotted in Fig. 4, show that for a given critical

shear stress, τ_m , the indentation pressure at failure depends on the distance from the edge to the initiating crack, and that for a given distance to the initiating crack, the dependence of $\ln(S_0/S_{01})$ on $\ln(2l)$ is essentially the same as that obtained by the previous calculations.

Relaxation of the elastic moduli

One of the features of the brittle to ductile transition in deformation behaviour is a corresponding relaxation of the elastic moduli (Traetteberg *et al.* 1975; Sinha 1978, 1988). The relaxation of Young's modulus begins for a strain rate of about 10^{-1} s^{-1} , and the value of the effective modulus decreases continuously with decreasing strain rate well into the ductile region of behaviour (i.e., $\dot{\epsilon} < 10^{-4} \text{ s}^{-1}$).

It can be seen from [2] to [6] inclusive that the stress imposed by the indenter is independent of the elastic moduli, except for Poisson's ratio through its effect on β . Gold (1958, 1990) found for type-S2 columnar-grained ice a correlation between the effective elastic modulus and the Poisson's ratio given by $\nu = 0.88 - 0.59E/E_0$, where E is the effective elastic modulus and E_0 is the true Young's modulus, taken to be 10 GPa (see Michel and Ramseier (1971) for definition of ice types). This correlation is consistent with the experimental results presented by Sinha (1988). As the effective elastic modulus decreases, the effective value of Poisson's ratio increases. It is equal to 0.5 for an elastic modulus of 6.4 GPa. This value is associated with a strain rate of about 10^{-5} s^{-1} for a temperature of -10°C . These figures are for columnar-grained, S2 ice, but a similar behaviour can be expected for the other types of ice.

Equations [2] and [3] show that for plane strain conditions β decreases from 0.0935 to 0 as the Poisson's ratio increases from 0.3 to 0.5. It must be kept in mind, however, that for ice these are the effective values of Poisson's ratio, and the change in its value is due primarily to the delayed elastic or time-dependent reversible component of the deformation. The size and the direction of this component will depend on the relation between the stress field and the type of ice. It will probably cause increased anisotropy in the total deformation of the ice, and the validity of using effective values of Poisson's ratio in [5]–[7] for these conditions must still be demonstrated. For the purpose of this discussion, however, it will be assumed that the equations can be used in the range of strain rates for which β decreases from a value of 0.0935, or larger, to 0.

The normalized maximum shear stresses, $F_r(\alpha, \theta, 0.5)$, for $\beta = 0$, $\nu = 0.5$, and $\alpha = 0.1$ and 0.01 are shown in Fig. 5. The normalized maximum $F_r(\alpha, \theta, 0.3)$ for $\beta = 0.0935$, $\nu = 0.3$, and $\alpha = 0.01$ is also shown for comparison. It can be seen that the change in β has resulted in a significant change in the stress distribution. For $\beta = 0$, the maximum value of the shear stress for a given value of α now lies on $\theta_m = 90^\circ$, and is smaller than the maximum value for the fully elastic condition. Along the interface between the indenter and the ice, $S_{xx} = S_{yy}$; and the shear stress, S_{xy} , has been reduced to zero. For $|x| > l$ and $y = 0$, $S_{xx} = S_{yy} = S_{xy} = 0$.

The line defined by $\sigma_2/\sigma_1 = 0.1$ and $\beta = 0$ is shown in Fig. 3. It lies within $\pm 3^\circ$ of $\theta = 70^\circ$ out to $\alpha = 0.8$. The zone of maximum shear, defined by $\theta_m \pm 5^\circ$, now lies wholly within the region for which $\sigma_2/\sigma_1 > 0.1$ out to $\alpha > 1.0$. It would be expected that the reduction of the maximum shear stress, and the increase in the minor principal stress in the region of maximum shear, would reduce the probability of crack initiation

and propagation. Assume, however, that when β just equals zero, a failure initiating crack forms in the high shear stress region near an edge along the line $\theta = 90^\circ$, and is propagated by a shear stress given by the condition $P_0 = 1 \text{ MN} \cdot \text{m}^{-1}$, $2l = 1 \text{ m}$, $r_c = 10^{-2} \text{ m}$, and $\nu = 0.3$. The indentation pressures at the indenter–ice interface at failure for indenter widths of 0.1, 1, 10, and 100 m were calculated using this reference stress and the position of the initiating crack at 10^{-2} , 2×10^{-2} , and $4 \times 10^{-2} \text{ m}$ from the edge. The results are presented in Fig. 6. Shown also is the dependence of S_0/S_{01} on the indenter width given in Fig. 4 for the reference condition. For all of the conditions considered, there is an increase in S_0 over that for the fully elastic condition, and the slope defined by the plotted points for a given crack position is a little smaller. Although the application of elastic theory has probably been extended beyond its valid range, the calculations do indicate that relaxation of the elastic moduli would cause the indentation pressure at failure to increase, but that this relaxation has little effect on the slope of the dependence of $\ln(S_0/S_{01})$ on $\ln(2l)$.

Calibration of the model and comparison with observations

Information has been published on the indentation pressures at failure for rigid, flat indentors impressed on the edge of ice plates. The thickness of the plates was relatively small compared with their length and breadth. If the indenter fully constrains the ice during indentation, the stress distribution in the ice would change from almost plane strain near the indenter to plane stress some distance away from it.

It was assumed that the critical shear stress for crack propagation, τ_m , was that for the plane strain condition and for $r_c = 10^{-2} \text{ m}$, $2l = 1 \text{ m}$, $\nu = 0.3$, and $P_0 = 1 \text{ MN} \cdot \text{m}^{-1}$. The indentation pressure for propagation of a crack initiated 10^{-2} m from the edge of the indenter was determined for the plane stress condition and for indenter widths of 0.1, 1.0, 10, and 100 m, using this value for τ_m . For these calculations, β was determined using [2] and [4] and $\nu = 0.3$. The resulting dependence of $\ln(S_0/S_{01})$ on $\ln(2l)$ is shown in Fig. 6. Again, the dependence of the slope of $\ln(S_0/S_{01})$ on $\ln(2l)$ is essentially the same, as found for the previous calculations. The indentation pressures for failure are only about 5% smaller than the ones for the corresponding plane strain condition. As the dependence of $\ln(S_0/S_{01})$ on $\ln(2l)$ is essentially the same for the plane strain and plane stress conditions, the results for plane strain will be used in the following calibration procedure.

It has been observed that the maximum indentation pressure for a given width of indenter depends on the rate of indentation, much in the same way as the compressive strength of ice depends on strain rate. If the rate of indentation is sufficiently high, brittle failure occurs; and the indentation pressure appears to be relatively constant and independent of the rate. As the indentation rate decreases from a relatively critical value, the indentation pressure increases to a maximum (Michel and Toussaint 1977; Sanderson 1988).

Michel and Toussaint give information on the indenter width and the indentation rate for the tests they carried out on 0.81 m square plates made from type-S2 columnar-grained ice. The indentors were applied to the edge of the plate so that the load direction was perpendicular to the long direction of the columnar grains. One of their tests appeared to be in the brittle range of behaviour, and gave a value of $S_0 = 13.5 \text{ MPa}$ for an indenter width of $2l = 50.8 \text{ mm}$. Another test appeared

FIG. 5. Normalized maximum shear stress at distance $r = \alpha_l$ for $\alpha_l = 0.1$ and 0.01 from edge $(l, 0)$ for Poisson's ratio $= 0.5$, and $\alpha_l = 0.01$ for Poisson's ratio $= 0.3$.

FIG. 6. Normalized indentation pressure to cause failure under an indenter of width $2l$ due to a crack formed at the given position r . The crack is propagated immediately by a shear stress equal to the maximum value for $r_c = 10^{-2}$ m, $2l = 1$ m, $\nu = 0.3$, indentation pressure $= S_{01}$, and plane strain conditions.

to be at the maximum of the dependence of S_0 on the rate of indenter penetration. It gave a value of $S_0 = 22.4$ MPa for the same width of indenter. These values were used to locate the lines corresponding to $r_c = 10^{-2}$ m and $\beta = 0.0935$ (brittle) and 0 (ductile) in Fig. 7. Shown also in the figure are the maximum indentation pressures for given indenter widths, as reported by several investigators.

FIG. 7. The dependence of indentation pressure to cause failure under an indenter of width $2l$ for $\beta = 0.0935$ (brittle condition) and $\beta = 0$ (ductile condition). Shown also are field and laboratory measurements of maximum indentation pressures for flat indentors penetrating ice covers of various thickness.

When interpreting the information given in this figure, it must be kept in mind that the measurements were made for various rates of penetration, thicknesses of ice, grain sizes, and temperatures. Michel and Blanchet (1983), Timco (1986), and Sodhi and Nakazawa (1988) worked with ice covers less

than 10 cm thick, and usually less than 5 cm, floating on water. Their ice surface temperatures were greater than -3°C . Failure, particularly for the wider indentors and thinner ice, was often associated with bending. Croasdale *et al.* (1977) worked with natural ice covers, and for 22 of their 25 tests the ice thickness was between 80 and 99 cm thick. Taylor (1981) worked with natural ice covers with thickness in the range of 16.5–32 cm. Both Croasdale *et al.* and Taylor applied their indentors to the edge of rectangular holes cut into the ice. The width of each hole was not much larger than that of the indenter. This condition must have applied a constraint that would not be present for an indenter impressed on a long straight edge. Michel and Toussaint (1977) used ice plates of thicknesses 2.54, 5.08, and 10.16 cm, maintained at a temperature of -10°C .

Some form of crushing was reported for almost all the results presented in Fig. 7. Crushing, however, was often associated with other modes of failure, e.g., spalling, radial cracking, circumferential cracking, and bending. In spite of all the modes of failure associated with the observed indentation pressures, the maximum values for similar ice conditions and temperatures show a decrease with increase in the width of indenter, which is consistent with that predicted from elastic theory.

Fracture toughness and the brittle to ductile transition

It was assumed that fracture occurs under an indenter when a crack initiated at a site is propagated by the shear stress at that location. This implies a mode II crack propagation and a stress intensity factor, K_{II} , given by

$$[13] \quad K_{II} = \tau\sqrt{\pi a}$$

where $2a$ is the crack width and τ is the shear stress parallel to the surface of the crack and perpendicular to its edge. It is appreciated that the problem is one of mixed-mode crack propagation and that the conditions for propagation are determined by the total biaxial stress field. The small amount of information on the critical value of K_{II} (the value of the stress intensity factor for the minimum value of shear stress to cause crack propagation in shear) indicates that it is about equal to the critical value of the factor, K_I , for mode I propagation due to a tensile stress at the crack tip perpendicular to the plane of the crack (Urabe and Inoue 1987). (For a description of fracture mechanics and a definition of mode I and mode II crack propagation, see Knott (1973).) The value of the apparent stress intensity factor, K_Q , for the indentation problem, assumed to be given by

$$[14] \quad K_Q = \tau\sqrt{\pi a}$$

should be close, therefore, to the corresponding values for K_I and K_{II} .

The maximum shear stress, τ_m , was determined for $2l = 50.8$ mm, $r_c = 10^{-2}$ m, $\theta = \theta_m$, and $S_0 = 13.5$ MPa ($\beta = 0.0935$) and 22.5 MPa ($\beta = 0$). The corresponding values for K_Q were calculated using [14] and $2a = 3.0$ mm and are presented in Table 1. For the fully brittle condition ($\beta = 0.0935$), the calculated value is larger than the measured critical values for mode I and mode II propagation, which are about $100 \text{ kPa} \cdot \text{m}^{-1/2}$ (Dempsey 1989; Urabe and Inoue 1987). A value in better agreement with the accepted critical values could be obtained by assuming either a smaller crack

TABLE 1. Calculated values of the apparent stress intensity factor, K_Q ($\text{kPa} \cdot \text{m}^{-1/2}$)

	Indentation	Simple compression
Brittle	380	172
Ductile	572	240

size or that the initial crack forms further from the edge, i.e., a smaller value for τ_m .

It is of interest to apply the same criterion to the crushing failure of ice in simple compression. A reasonable value for the crushing strength of type-S2 ice at -10°C and $\dot{\epsilon} > 10^{-2} \text{ s}^{-1}$ is 5 MPa; and for the maximum value which occurs at $\dot{\epsilon} \approx 5 \times 10^{-3} \text{ s}^{-1}$, 7 MPa (Michel and Toussaint 1977, Fig. 10). These values give for the maximum shear stress 2.5 and 3.5 MPa, respectively. The values for K_Q , using these values for τ_m in [14] and a crack width, $2a$, of 3.0 mm, are given in Table 1. It can be seen that they are closer to the critical values and are in reasonable agreement with those calculated for the indentation problem.

There is another possible reason, in addition to crack size and position, why the calculated apparent stress intensity factor for brittle failure during indentation is larger than the critical values of K_I and K_{II} . The requirement of a stress condition that will cause crack propagation is necessary but not sufficient for failure. Observations have shown that a strain of about 5×10^{-4} or larger is necessary to nucleate cracks in ice subject to simple compression. The amount of strain increases with increasing minor compressive stress (Kalifa 1988). For ice in simple compression strained at a rate greater than 10^{-2} s^{-1} ($E \approx 10$ GPa), a strain of 5×10^{-4} corresponds to a compressive stress of 5 MPa. A crack of the appropriate size may not be initiated, therefore, before the maximum shear stress exceeds the value that would cause it to propagate. This indicates that the necessary and sufficient conditions for the fully brittle failure of ice during indentation are a critical amount of shear strain imposed at a sufficiently high rate of strain.

On the other hand, as the strain rate decreases through the brittle to ductile transition region, the apparent stress intensity factor for propagation increases (Nixon and Schulson 1986), and at some strain rate it should equal that imposed by the applied stress. For brittle failure to occur at still lower strain rates, the load for failure must increase because of the associated increase in the apparent stress intensity factor for crack propagation. It would be expected that the fully ductile condition would be established in ice when the apparent stress intensity factor required to propagate cracks that form is greater than the actual value that can be induced at their tips at the imposed strain rate.

The work of Cole (1987) shows that this condition probably depends also on grain size. He found that the maximum compressive strength in the brittle to ductile range of strain rates was significantly larger for small grained ice than for large grained ice; and that the large grained ice tended to respond in a brittle manner to a lower strain rate than the small grained ice. This behaviour is consistent with the assumption that the size of the initiating crack is of the same order as the grain size, and with [14], i.e., the larger the size of the crack, the smaller is the shear stress for propagation for a given value of the apparent stress intensity factor.

Dependence of strain rate on indentation velocity

The example that has been used to demonstrate features of the brittle to ductile transition in ice during indentation has an unfortunate shortcoming: the deflection of the interface under an indenter impressed on a semi-infinite body is infinite. Real ice bodies, however, are finite. Indentation, furthermore, is measured from some position that is assumed to be fixed, but need not be so. The point to be made is that the dependence of strain on the amount of indentation, or strain rate on indentation rate, depends very much on the size of the body of ice and the conditions on boundaries away from the area of contact with the indenter.

For a flat indenter applied to a semi-infinite body, the indentation is given by the following integral equation:

$$[15] \quad \frac{4\pi\mu v_0}{P_0(1+x)} = \int_1^\infty \frac{\cos(\beta \ln \frac{t+1}{t-1})}{\sqrt{(t+1)(t-1)}} dt$$

where v_0 = indentation, μ = shear modulus, and $t = x/l$. It can be seen from [15] that not only does the value of the integral become very large as t becomes very large, it oscillates also as t approaches 1. Because of the small value of β , however, oscillations of the integrand only occur very close to the edge of the indenter and well within the zone for which elastic theory can no longer be expected to apply because of high stresses.

Assume that a reasonable measure of the indentation is given by integrating [15] over the range of $1.001 \leq t \leq 100$. With this assumption,

$$[16] \quad \frac{P_0}{2\pi} = \frac{0.38v_0\mu}{1+x}$$

The maximum shear strain, θ_m , at point α is given by

$$[17] \quad \gamma_m = \frac{\tau_m}{\mu} = \frac{P_0}{2\pi l\mu} F_\tau(\alpha, \theta_m, \nu)$$

Substituting for $P_0/2\pi$ from [16] and neglecting the time dependence of χ and $F_\tau(\alpha, \theta, \nu)$ gives

$$[18] \quad \dot{\gamma}_m = \frac{0.38}{(1+x)l} \dot{v}_0 F_\tau(\alpha, \theta_m, \nu)$$

Table 2 presents the values of $\dot{v}_0$ for $r_c = 10^{-2}$ m; $2l = 0.1, 1, 10$, and 100 m; and $\dot{\gamma}$ equals to $1.3 \times 10^{-2} \text{ s}^{-1}$ ($\nu = 0.3$) and $1.5 \times 10^{-3} \text{ s}^{-1}$ ($\nu = 0.5$). The values for brittle behaviour are smaller than those given by Michel and Tous-saint (1977), by a factor of about 5. This degree of agreement is probably satisfactory, considering the assumption of ductile behaviour underlying the calculations of Michel and Tous-saint, and that the relation between strain rate and indentation rate depends on the size of the ice body and the conditions of restraint at its edges.

Discussion

One of the principal purposes of this paper has been to demonstrate by a relatively simple problem that the brittle to ductile transition for ice during indentation can be usefully explored using equations derived from the theory of elasticity. In the process, it has been shown that the problem gives results that are consistent with the dependence of indentation pressure on indenter width that has been observed for laboratory and field measurements. It can be shown that, unlike the depen-

TABLE 2. Indentation rate to give a maximum shear rate of $1.3 \times 10^{-2} \text{ s}^{-1}$, 10^{-2} m from the edge of indenter ($\nu = 0.3$); and $1.5 \times 10^{-3} \text{ s}^{-1}$ ($\nu = 0.5$), for various widths of indenter

Indenter width (m)	$\dot{v}_0 \text{ (m} \cdot \text{s}^{-1}\text{)}$	
	$\nu = 0.3$	$\nu = 0.5$
0.1	2.67×10^{-3}	2.46×10^{-4}
1.0	7.70×10^{-3}	7.89×10^{-4}
10.0	2.13×10^{-2}	2.49×10^{-3}
100.0	5.80×10^{-2}	7.89×10^{-3}

dence of strain rate on indentation rate, this dependence depends primarily on the boundary conditions at the indenter.

The dependence of τ_m on S_{xx} , S_{yy} , and S_{xy} is given by

$$[19] \quad \tau_m \left(\frac{2\pi l}{P_0} \right) = \sqrt{\left(\frac{S_{xx} - S_{yy}}{2} \right)^2 + S_{xy}^2}$$

If the failure initiating crack forms at a distance r_c from edge 1, and $r_c \ll 2l$, then $\alpha_2 \approx 2$. Substituting this value for α_2 in [5]–[7] gives

$$[20] \quad \tau_m \left(\frac{2\pi l}{P_0} \right) = F_\tau(\alpha_c, \theta_m, \nu) \approx \frac{1}{\sqrt{2\alpha_1}} F'_\tau(\alpha_c, \theta_m, \nu)$$

Consider failure under indentors of width $2l_1$, and $2l_2$, occurring for the same critical maximum shear stress, τ_m , and distance of initiating crack from the edge, r_c . From [20], the ratio of the indentation pressures at failure is given by

$$[21] \quad \frac{S_2}{S_1} = \sqrt{\frac{l_1}{l_2} \frac{F'_{\tau 2}}{F'_{\tau 1}}}$$

$F'_{\tau 1}$ and $F'_{\tau 2}$ can be determined for each value of α , from the calculated values of F_τ . For indenter widths of 1 and 100 m ($\alpha_1 = 2 \times 10^{-2}$ and 2×10^{-4}), $r_c = 10^{-2}$ m, $\nu = 0.3$, and θ_m , the angular coordinate for the maximum shear stress for the given α_1 , $F'_\tau = 1.25$ and 1.66 , respectively. Substituting these values into [21] gives for the ratio of the indentation pressures

$$[22] \quad \frac{S_2}{S_1} = \left(\frac{1}{100} \right)^{0.56}$$

For $\nu = 0.5$, the value of F'_τ for the 1 m indenter is equal to that for the 100 m indenter, and so the exponent in [22] is 0.5. These values agree with the slopes of the corresponding two lines in Fig. 6. The foregoing calculation demonstrates that for the rigid rectangular indenter, the slope of the dependence at failure of $\ln S_0$ on $\ln 2l$ is determined almost entirely by the nature of the singularities in the stress field induced by the edges of the indenter.

This study has identified the initiation of failure as an important event in the transition from brittle to ductile behaviour. There are two interesting boundary conditions that define the range of response that can be expected after the propagation of the initial crack. One is the condition of an indenter penetrating the ice at a fixed rate. When the failure initiating crack propagates, there will be an associated drop in load to the level that can be carried by the still intact ice. If the load is not great enough to fail the ice, it will increase until another crack is initiated than can propagate. Failure will progress

from the edge across the width of the indenter.

The second condition is that of an indenter to which a given load, P_0 , is applied. When the fracture initiating crack occurs, the load is suddenly transferred to a smaller area with a corresponding increase in stress. This is the condition that would be expected for loading systems of the type used to obtain the results presented in Fig. 7. The elasticity of such indenter supports would cause the load to remain essentially constant during the initial failure process. Failure would again progress across the face of the indenter, but it would do so faster for the same nominal penetration rate than it would for the constant penetration rate condition.

The criteria for failure that have been assumed imply a dependence on grain size. The maximum indentation pressure in the transition from brittle to ductile behaviour for a given indenter width also depends on grain size. Grain size, therefore, is an important factor to be considered in the testing and calibration of the model. It must contribute, as well, to some of the variability that is observed in the measurements of indentation pressures in the laboratory and in the field.

Nothing has been said about the conditions very close to the edge of the indenter. Normally it is assumed that this highly stressed region is in a plastic state, and is hypothetically isolated from the surrounding elasticity behaving material. The model implies, however, that the stress singularity at an edge plays a major role in determining the response of ice to indentation loads. It is of interest to speculate, therefore, on what happens in the highly stressed region up to, through, and beyond the brittle to ductile transition.

At the high indentation rates associated with the fully brittle range, the very large stresses at the edge would be imposed on individual ice crystals at the surface of the ice, and would be relieved by microfractures and slip essentially confined to individual grains. Dislocation-controlled slip would become relatively more important as the stress relieving mechanism as the indentation rate is decreased. At the strain rate for which β just becomes equal to zero, it could probably be assumed that the deformation response in the highly stressed material at an edge is fully plastic. For strain rates associated with ductile behaviour, it would be expected that the plastic region would grow with the amount of indentation. This would modify the stress state until the condition of fully plastic indentation is attained, for which $S_{xx} = S_{yy} = \text{constant}$ and $S_{xy} = 0$ in the ice at the indenter face.

The results of the present study indicate that the theory of elasticity could be usefully applied to other geometries of contact between ice and another solid. It would be of interest to explore this application for multiple contact problems, using the theory of superposition. This would require formulating the contribution to the stress field of linear and curved elements of ice impacting an extended solid surface. Having this capability would allow calculations to be made for the irregular contact conditions normally found during the interaction between ice and structures. It should be possible to include in the calculations the effects of external stress fields and the dynamic behaviour of the structure.

Conclusions

Elastic theory combined with information on crack initiation and linear elastic fracture mechanics provides a reasonable basis for analysis of the brittle to ductile transition in the deformation behaviour of ice. It predicts that a flat indenter pene-

trating a flat edge of a large ice plate, at a rate that imposes in the ice maximum strain rates in the range associated with the brittle to ductile transition, will cause indentation pressures at failure that vary inversely with the square root of the width of the indenter and that increase to a maximum value with decreasing strain rate. Brittle failure of flaw-free ice during indentation requires that the shear strain exceed the critical value that is necessary for cracks to be initiated by the imposed stress field. Failure will occur if the stresses on crack are sufficiently large to cause it to propagate at the time it is initiated. With further development, the approach could prove suitable for determining ice forces against structures during failure by crushing.

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List of symbols

$\phi(Z)$	stress function
Z	$x + iy$ = position in complex plane
P_0	load per unit length
S_0	indentation pressure
$2l$	width of indenter
$2a$	crack width
r	radial position
θ	angular position
α	r/l
t	x/l
Z_1	$\beta \ln(r_2/r_1) + (\theta_2 + \theta_1)/2$
Z_2	$\beta \ln(r_2/r_1) - (\theta_2 + \theta_1)/2$
Z_3	$\beta \ln(r_2/r_1) + 3(\theta_2 + \theta_1)/2$
A	$1 + \alpha_1 \cos \theta_1$
B	$\alpha_1 \sin \theta_1 + 2\beta$
μ	shear modulus
ν	Poisson's ratio
χ	$3 - 4\nu$ (plane strain); $(3 - \nu)/1 + \nu$ (plane stress)
β	$(\ln \chi)/2\pi$
τ	shear stress
τ_m	maximum shear stress
γ	shear strain
$F_r(\alpha, \theta, \nu)$	normalized maximum shear stress
K_I	critical stress intensity factor for mode I propagation
K_{II}	critical stress intensity factor for mode II propagation
K_Q	effective critical stress intensity factor
ν_0	indentation