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π/2-Angle Yao Graphs are Spanners

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Abstract. We show that the Yao graph $Y_4$ in the $L_2$ metric is a spanner with stretch factor $8\sqrt{2}(29 + 23\sqrt{2})$.

1 Introduction

Let $V$ be a finite set of points in the plane and let $G = (V, E)$ be the complete Euclidean graph on $V$. We will refer to the points in $V$ as nodes, to distinguish them from other points in the plane. The Yao graph [7] with an integer parameter $k > 0$, denoted $Y_k$, is defined as follows. Any $k$ equally-separated rays starting at the origin define $k$ cones. Pick a set of arbitrary, but fixed cones. We can now translate the cones to each node $u \in V$. In each cone, pick a shortest edge $uv$, if there is one, and add to $Y_k$ the directed edge $\rightarrow uv$. Ties are broken arbitrarily. Note that the Yao graph differs from the $\Theta$-graph in how the shortest edge is chosen. While the Yao graph chooses the shortest edge in terms of the Euclidean distance, the $\Theta$-graph chooses the shortest edge as the one that has the shortest distance to $u$ after being projected to the bisector of the cone. Most of the time we ignore the direction of an edge $uv$; we refer to the directed version $\rightarrow uv$ only when its origin $(u)$ is important and unclear from the context. We will distinguish between $Y_k$, the Yao graph in the Euclidean $L_2$ metric, and $Y_k^\infty$, the Yao graph in the $L_\infty$ metric. Unlike $Y_k$ however, in constructing $Y_k^\infty$ ties are broken by always selecting the most counterclockwise edge; the reason for this choice will become clear in Section 2.

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For a given subgraph $H \subseteq G$ and a fixed $t \geq 1$, $H$ is called a $t$-spanner for $G$ if, for any two nodes $u, v \in V$, the shortest path in $H$ from $u$ to $v$ is no longer than $t$ times the length of $uv$. The value $t$ is called the dilation or the stretch factor of $H$. If $t$ is constant, then $H$ is called a length spanner, or simply a spanner.

The class of graphs $Y_k$ has been much studied. Bose et al. [2] showed that, for $k \geq 9$, $Y_k$ is a spanner with stretch factor $\frac{1}{\cos \frac{\pi}{k} - \sin \frac{\pi}{k}}$. In [1] we improve the stretch factor and show that, in fact, $Y_k$ is a spanner for any $k \geq 7$. Recently, Molla [5] showed that $Y_2$ and $Y_3$ are not spanners, and that $Y_4$ is a spanner with stretch factor $4(2 + \sqrt{2})$, for the special case when the nodes in $V$ are in convex position (see also [3]). The authors conjectured that $Y_4$ is a spanner for arbitrary point sets. In this paper, we settle their conjecture and prove that $Y_4$ is a spanner with stretch factor $8\sqrt{2}(29 + 23\sqrt{2})$.

The paper is organized as follows. In Section 2, we prove that the graph $Y_4^\infty$ is a spanner with stretch factor 8. In Section 3 we establish several properties for the graph $Y_4$. Finally, in Section 4, we use the properties of Section 3 to prove that, for every edge $ab$ in $Y_4^\infty$, there exists a path between $a$ and $b$ in $Y_4$ not much longer than the Euclidean distance between $a$ and $b$. By combining this with the result of Section 2, it follows that $Y_4$ is a spanner.

2 $Y_4^\infty$ in the $L_\infty$ Metric

In this section we focus on $Y_4^\infty$, which has a nicer structure compared to $Y_4$. First we prove that $Y_4^\infty$ is a plane graph. Then we use this property to show that $Y_4^\infty$ is an 8-spanner. To be more precise, we prove that for any two nodes $a$ and $b$, the graph $Y_4^\infty$ contains a path between $a$ and $b$ whose length (in the $L_\infty$-metric) is at most $8|ab|_\infty$.

We need a few definitions. We say that two edges $ab$ and $cd$ properly cross (or cross, for short) if they share a point other than an endpoint ($a, b, c$ or $d$); we say that $ab$ and $cd$ intersect if they share a point (either an interior point or an endpoint). Let $Q_1(a), Q_2(a), Q_3(a)$ and $Q_4(a)$ be the four quadrants at $a$, as in

![Diagram](image-url)

**Fig. 1.** (a) Definitions: $Q_i(a), P_i(a)$ and $S(a, b)$. (b) Lemma 1: $ab$ and $cd$ cannot cross.
Figure 1a. Let \( P(a) \) be the path that starts at point \( a \) and follows the directed Yao edges in quadrant \( Q_i \). Let \( P_i(a,b) \) be the subpath of \( P(a) \) that starts at \( a \) and ends at \( b \). Let \( |ab|_\infty \) be the \( L_\infty \) distance between \( a \) and \( b \). Let \( sp(a,b) \) denote a shortest path in \( Y_4^\infty \) between \( a \) and \( b \). Let \( S(a,b) \) denote the open square with corner \( a \) whose boundary contains \( b \), and let \( \partial S(a,b) \) denote the boundary of \( S(a,b) \). These definitions are illustrated in Figure 1a. For a node \( a \in V \), let \( x(a) \) denote the \( x \)-coordinate of \( a \) and \( y(a) \) denote the \( y \)-coordinate of \( a \).

**Lemma 1.** \( Y_4^\infty \) is a plane graph.

**Proof.** The proof is by contradiction. Assume the opposite. Then there are two edges \( \overrightarrow{ab}, \overrightarrow{cd} \in Y_4^\infty \) that cross each other. Since \( \overrightarrow{ab} \in Y_4^\infty \), \( S(a,b) \) must be empty of nodes in \( V \), and similarly for \( S(c,d) \). Let \( j \) be the intersection point between \( ab \) and \( cd \). Then \( j \in S(a,b) \cap S(c,d) \), meaning that \( S(a,b) \) and \( S(c,d) \) must overlap. However, neither square may contain \( a,b,c \) or \( d \). It follows that \( S(a,b) \) and \( S(c,d) \) coincide, meaning that \( c \) and \( d \) lie on \( \partial S(a,b) \) (see Figure 1b). Since \( cd \) intersects \( ab \), \( c \) and \( d \) must lie on opposite sides of \( ab \). Thus either \( ac \) or \( ad \) lies counterclockwise from \( ab \). Assume without loss of generality that \( ac \) lies counterclockwise from \( ab \); the other case is identical. Because \( S(a,c) \) coincides with \( S(a,b) \), we have that \( |ac|_\infty = |ab|_\infty \). In this case however, \( Y_4^\infty \) would break the tie between \( ac \) and \( ab \) by selecting the most counterclockwise edge, which is \( \overrightarrow{ac} \). This contradicts that \( \overrightarrow{ab} \in Y_4^\infty \). \( \square \)

**Theorem 1.** \( Y_4^\infty \) is an 8-spanner in the \( L_\infty \) metric space.

**Proof.** We show that, for any pair of points \( a,b \in V \), \(|sp(a,b)|_\infty < 8|ab|_\infty \). The proof is by induction on the pairwise distance between the points in \( V \). Assume without loss of generality that \( b \in Q_1(a) \), and \( |ab|_\infty = |x(b) - x(a)| \). Consider the case in which \( ab \) is a closest pair of points in \( V \) (the base case for our induction). If \( ab \in Y_4^\infty \), then \(|sp(a,b)|_\infty = |ab|_\infty \). Otherwise, there must be \( ac \in Y_4^\infty \), with \(|ac|_\infty = |ab|_\infty \). But then \(|bc|_\infty < |ab|_\infty \) (see Figure 2a), a contradiction.

![Fig. 2.](image-url)

(a) Base case. (b) \( \triangle abc \) empty (c) \( \triangle abc \) non-empty, \( P_{ar} \cap P_{r(b)} = \{j\} \) (d) \( \triangle abc \) non-empty, \( P_{ar} \cap P_{r(b)} = \emptyset \), \( e \) above \( r \) (e) \( \triangle abc \) non-empty, \( P_{ar} \cap P_{r(b)} = \emptyset \), \( e \) below \( r \).
Assume now that the inductive hypothesis holds for all pairs of points closer than \(|ab|_\infty\). If \(ab \in Y_4^\infty\), then \(|sp(a, b)|_\infty = |ab|_\infty\) and the proof is finished. If \(ab \notin Y_4^\infty\), then the square \(S(a, b)\) must be nonempty.

Let \(A\) be the rectangle \(ab'ba'\) as in Figure 2b, where \(ba'\) and \(bb'\) are parallel to the diagonals of \(S\). If \(A\) is nonempty, then we can use induction to prove that \(|sp(a, b)|_\infty \leq 8|ab|_\infty\) as follows. Pick \(c \in A\) arbitrary. Then \(|ac|_\infty + |cb|_\infty = |x(c) - x(a)| + |x(b) - x(c)| = |ab|_\infty\), and by the inductive hypothesis \(sp(a, c) \oplus sp(c, b)\) is a path in \(Y_4^\infty\) no longer than \(8|ac|_\infty + 8|cb|_\infty = 8|ab|_\infty\); here \(\oplus\) represents the concatenation operator. Assume now that \(A\) is empty. Let \(c\) be at the intersection between the line supporting \(ba'\) and the vertical line through \(a\) (see Figure 2b). We discuss two cases, depending on whether \(\triangle abc\) is empty of points or not.

**Case 1:** \(\triangle abc\) is empty of points. Let \(ad \in P_1(a)\). We show that \(P_4(d)\) cannot contain an edge crossing \(ab\). Assume the opposite, and let \(st \in P_4(d)\) cross \(ab\). Since \(\triangle abc\) is empty, \(s\) must lie above \(bc\) and \(t\) below \(ab\), therefore \(|st|_\infty \geq |y(s) - y(t)| > |y(s) - y(b)| = |ab|_\infty\), contradicting the fact that \(st \in Y_4^\infty\). It follows that \(P_4(d)\) and \(P_2(b)\) must meet in a point \(i \in P_4(d) \cap P_2(b)\) (see Figure 2b). Now note that \(|P_4(d, i) \oplus P_2(b, i)|_\infty \leq |x(d) - x(b)| + |y(d) - y(b)| < 2|ab|_\infty\). Thus we have that \(|sp(a, b)|_\infty \leq |ad \oplus P_4(d, i) \oplus P_2(b, i)|_\infty < |ab|_\infty + 2|ab|_\infty = 3|ab|_\infty\).

**Case 2:** \(\triangle abc\) is nonempty. In this case, we seek a short path from \(a\) to \(b\) that does not cross to the underside of \(ab\), to avoid oscillating paths that cross \(ab\) arbitrarily many times. Let \(r\) be the rightmost point that lies inside \(\triangle abc\). Arguments similar to the ones used in Case 1 show that \(P_3(r)\) cannot cross \(ab\) and therefore it must meet \(P_1(a)\) in a point \(i\). Then \(P_{ar} = P_1(a, i) \oplus P_3(r, i)\) is a path in \(Y_4^\infty\) of length

\[
|P_{ar}|_\infty < |x(a) - x(r)| + |y(a) - y(r)| < |ab|_\infty + 2|ab|_\infty = 3|ab|_\infty.
\]  
(1)

The term \(2|ab|_\infty\) in the inequality above represents the fact that \(|y(a) - y(r)| \leq |y(a) - y(c)| \leq 2|ab|_\infty\). Consider first the simpler situation in which \(P_2(b)\) meets \(P_{ar}\) in a point \(j \in P_2(b) \cap P_{ar}\) (see Figure 2c). Let \(P_{ar}(a, j)\) be the subpath of \(P_{ar}\) extending between \(a\) and \(j\). Then \(P_{ar}(a, j) \oplus P_2(b, j)\) is a path in \(Y_4^\infty\) from \(a\) to \(b\), therefore \(|sp(a, b)|_\infty \leq |P_{ar}(a, j) \oplus P_2(b, j)|_\infty < 2|y(j) - y(a)| + |ab|_\infty \leq 5|ab|_\infty\).

Consider now the case when \(P_2(b)\) does not intersect \(P_{ar}\). We argue that, in this case, \(Q_1(r)\) may not be empty. Assume the opposite. Then no edge \(st \in P_2(b)\) may cross \(Q_1(r)\). This is because, for any such edge, \(|sr|_\infty < |st|_\infty\), contradicting \(st \in Y_4^\infty\). This implies that \(P_2(b)\) intersects \(P_{ar}\), again a contradiction to our assumption. This establishes that \(Q_1(r)\) is nonempty. Let \(rd \in P_1(r)\). The fact that \(P_2(b)\) does not intersect \(P_{ar}\) implies that \(d\) lies to the left of \(b\). The fact that \(r\) is the rightmost point in \(\triangle abc\) implies that \(d\) lies outside \(\triangle abc\) (see Figure 2d). It also implies that \(P_4(d)\) shares no points with \(\triangle abc\). This along with arguments similar to the ones used in case 1 show that \(P_4(d)\) and \(P_2(b)\) meet in a point \(j \in P_4(d) \cap P_2(b)\). Thus we have found a path

\[
P_{ab} = P_1(a, i) \oplus P_3(r, i) \oplus rd \oplus P_4(d, j) \oplus P_2(b, j)
\]  
(2)
extending from $a$ to $b$ in $Y_4^\infty$. If $|rd|_\infty = |x(d) - x(r)|$, then $|rd|_\infty < |x(b) - x(a)| = |ab|_\infty$, and the path $P_{ab}$ has length

$$|P_{ab}|_\infty \leq 2|y(d) - y(a)| + |ab|_\infty < 7|ab|_\infty. \quad (3)$$

In the above, we used the fact that $|y(d) - y(a)| = |y(d) - y(r)| + |y(r) - y(a)| < |ab|_\infty + 2|ab|_\infty$. Suppose now that

$$|rd|_\infty = |y(d) - y(r)|. \quad (4)$$

In this case, it is unclear whether the path $P_{ab}$ defined by (2) is short, since $rd$ can be arbitrarily long compared to $ab$. Let $e$ be the clockwise neighbor of $d$ along the path $P_{ab}$ (e and $b$ may coincide). Then $e$ lies below $a$, and either $de \in P_4(d)$, or $ed \in P_2(e)$ (or both). If $e$ lies above $r$, or at the same level as $r$ (i.e., $e \in Q_4(r)$, as in Figure 2d), then

$$|y(e) - y(r)| < |y(d) - y(r)| \quad (5)$$

This along with inequalities (4) and (5) implies $|re|_\infty > |y(e) - y(r)|$, which in turn implies $|re|_\infty = |x(e) - x(r)| \leq |ab|_\infty$, and so $|rd|_\infty \leq |ab|_\infty$. Then inequality (3) applies here as well, showing that $|P_{ab}|_\infty < 7|ab|_\infty$.

If $e$ lies below $r$ (as in Figure 2e), then

$$|ed|_\infty \geq |y(d) - y(e)| \geq |y(d) - y(r)| = |rd|_\infty. \quad (6)$$

Assume first that $ed \in P_2(e)$, or $|ed|_\infty = |x(e) - x(d)|$. In either case, $|ed|_\infty \leq |er|_\infty < 2|ab|_\infty$. This along with inequality (6) shows that $|rd|_\infty < 2|ab|_\infty$. Substituting this upper bound in (2), we get $|P_{ab}|_\infty \leq 2|y(d) - y(a)| + 2|ab|_\infty < 8|ab|_\infty$. Assume now that $ed \notin P_2(e)$, and $|ed|_\infty = |y(e) - y(d)|$. Then $ee' \in P_2(e)$ cannot go above $d$ (otherwise $|ed|_\infty < |ee'|_\infty$, contradicting $ee' \in P_2(e)$). This along with the fact $de \in P_4(d)$ implies that $P_2(e)$ intersects $P_{ar}$ in a point $k$. Redefine $P_{ab} = P_{ar}(a, k) \oplus P_4(e, k) \ominus P_4(e, j) \ominus P_2(b, j)$. Then $P_{ab}$ is a path in $Y_4^\infty$ from $a$ to $b$ of length $|P_{ab}|_\infty \leq 2|y(r) - y(a)| + |ab|_\infty \leq 5|ab|_\infty$. \ □

This theorem will be employed in Section 4.

### 3 Y₄ in the L₂ Metric

In this section we establish basic properties of $Y_4$. Due to space restrictions, some of these properties are stated without proofs. The proofs can be found in [1]. The ultimate goal of this section is to show that, if two edges in $Y_4$ cross, there is a short path between their endpoints (Lemma 8). We begin with a few definitions.

Let $Q(a, b)$ denote the infinite quadrant with origin at $a$ that contains $b$. For a pair of nodes $a, b \in V$, define recursively a directed path $P(a \to b)$ from $a$ to $b$ in $Y_4$ as follows. If $a = b$, then $P(a \to b) = \text{null}$. If $a \neq b$, there must exist $\overline{ac} \in Y_4$ that lies in $Q(a, b)$. In this case, define

$$P(a \to b) = \overline{ac} \oplus P(c \to b).$$
Recall that \( \oplus \) represents the concatenation operator. This definition is illustrated in Figure 3a. Fischer et al. [4] show that \( P(a \rightarrow b) \) is well defined and lies entirely inside the square centered at \( b \) whose boundary contains \( a \).

For any node \( a \in V \), let \( D(a, r) \) denote the open disk centered at \( a \) of radius \( r \), and let \( \partial D(a, r) \) denote the boundary of \( D(a, r) \). Let \( D[a, r] = D(a, r) \cup \partial D(a, r) \).

For any path \( P \) and any pair of nodes \( a, b \in P \), let \( P[a, b] \) be the subpath of \( P \) from \( a \) to \( b \). Let \( R(a, b) \) be the closed rectangle with diagonal \( ab \).

For a fixed pair of nodes \( a, b \in V \), define a path \( P_R(a \rightarrow b) \) as follows. Let \( e \in V \) be the first node along \( P(a \rightarrow b) \) that is not strictly interior to \( R(a, b) \). Then \( P_R(a \rightarrow b) \) is the subpath of \( P(a \rightarrow b) \) that extends between \( a \) and \( e \). In other words, \( P_R(a \rightarrow b) \) is the path that follows the \( Y_4 \) edges pointing towards \( b \), truncated as soon as it reaches \( b \) or leaves \( R(a, b) \). Formally, \( P_R(a \rightarrow b) = P(a \rightarrow b)[a, e] \). This definition is illustrated in Figure 3b. Our proofs will make use of the following two propositions.

**Proposition 1.** The sum of the lengths of crossing diagonals of a non-degenerate (necessarily convex) quadrilateral \( abcd \) is strictly greater than the sum of the lengths of either pair of opposite sides:

\[
|ac| + |bd| > |ab| + |cd|
|ac| + |bd| > |bc| + |da|
\]

**Proposition 2.** For any triangle \( \triangle abc \), the following inequalities hold:

\[
|ac|^2 \begin{cases} < |ab|^2 + |bc|^2, & \text{if } \angle abc < \pi/2 \\ = |ab|^2 + |bc|^2, & \text{if } \angle abc = \pi/2 \\ > |ab|^2 + |bc|^2, & \text{if } \angle abc > \pi/2 \end{cases}
\]

**Lemma 2.** For each pair of nodes \( a, b \in V \),

\[
|P_R(a \rightarrow b)| \leq |ab|\sqrt{2}
\] (7)

Furthermore, each edge of \( P_R(a \rightarrow b) \) is no longer than \( |ab| \).
Proof. Let $c$ be one of the two corners of $R(a, b)$, other than $a$ and $b$. Let $de \in \mathcal{P}_R(a \to b)$ be the last edge on $\mathcal{P}_R(a \to b)$, which necessarily intersects $\partial R(a, b)$ (note that it is possible that $e = b$). Refer to Figure 3b. Then $|de| \leq |db|$, otherwise $de$ could not be in $Y_4$. Since $db$ lies in the rectangle with diagonal $ab$, we have that $|db| \leq |ab|$, and similarly for each edge on $\mathcal{P}_R(a \to b)$. This establishes the latter claim of the lemma. For the first claim of the lemma, let $p = \mathcal{P}_R(a \to b)[a, d] \cap db$. Since $|de| \leq |db|$, we have that $|\mathcal{P}_R(a \to b)| \leq |p|$. Since $p$ lies entirely inside $R(a, b)$ and consists of edges pointing towards $b$, we have that $p$ is an $xy$-monotone path. It follows that $|p| \leq |ac| + |cb|$, which is bounded above by $|ab|\sqrt{2}$. \hfill $\square$

Lemma 3. Let $a, b, c, d \in V$ be four disjoint nodes such that $\overrightarrow{ab}, \overrightarrow{cd} \in Y_4$, $b \in Q_i(a)$ and $d \in Q_i(c)$, for some $i \in \{1, 2, 3, 4\}$. Then $ab$ and $cd$ cannot cross.

The next four lemmas (4–8) each concern a pair of crossing $Y_4$ edges, culminating (in Lemma 8) in the conclusion that there is a short path in $Y_4$ between a pair of endpoints of those edges.

Lemma 4. Let $a, b, c$ and $d$ be four disjoint nodes in $V$ such that $\overrightarrow{ab}, \overrightarrow{cd} \in Y_4$, and $ab$ crosses $cd$. Then (i) the ratio between the shortest side and the longer diagonal of the quadrilateral $abcd$ is no greater than $1/\sqrt{2}$, and (ii) the shortest side of the quadrilateral $abcd$ is strictly shorter than either diagonal.

Lemma 5. Let $a, b, c, d$ be four distinct nodes in $V$, with $c \in Q_1(a)$, such that (i) $\overrightarrow{ab} \in Q_1(a)$ and $\overrightarrow{cd} \in Q_3(c)$ are in $Y_4$ and cross each other, and (ii) $ad$ is a shortest side of quadrilateral $abcd$. Then $\mathcal{P}_R(a \to d)$ and $\mathcal{P}_R(d \to a)$ have a nonempty intersection.

Lemma 6. Let $a, b, c, d$ be four distinct nodes in $V$, with $c \in Q_1(a)$, such that (i) $\overrightarrow{ab} \in Q_1(a)$ and $\overrightarrow{cd} \in Q_3(c)$ are in $Y_4$ and cross each other, and (ii) $ad$ is a shortest side of quadrilateral $abcd$. Then $\mathcal{P}_R(d \to a)$ does not cross $ab$.

The next lemma relies on all of Lemmas 2–6.

Lemma 7. Let $a, b, c, d \in V$ be four distinct nodes such that $\overrightarrow{ab} \in Y_4$ crosses $\overrightarrow{cd} \in Y_4$, and let $xy$ be a shortest side of the quadrilateral $abcd$. Then there exist two paths $\mathcal{P}_x$ and $\mathcal{P}_y$ in $Y_4$, where $\mathcal{P}_x$ has $x$ as an endpoint and $\mathcal{P}_y$ has $y$ as an endpoint, with the following properties:

(i) $\mathcal{P}_x$ and $\mathcal{P}_y$ have a nonempty intersection.
(ii) $|\mathcal{P}_x| + |\mathcal{P}_y| \leq 3\sqrt{2}|xy|.$
(iii) Each edge on $\mathcal{P}_x \cup \mathcal{P}_y$ is no longer than $|xy|$.

Proof. Assume without loss of generality that $b \in Q_1(a)$. We discuss the following exhaustive cases:

1. $c \in Q_1(a)$, and $d \in Q_1(c)$. In this case, $ab$ and $cd$ cannot cross each other (by Lemma 3), so this case is finished.
2. \( c \in Q_1(a) \) and \( d \in Q_2(c) \), as in Figure 4a. Since \( ab \) crosses \( cd \), \( b \in Q_2(c) \).
Since \( \overrightarrow{ab} \in Y_4 \), \( |ab| \leq |ac| \). Since \( \overrightarrow{cd} \in Y_4 \), \( |cd| \leq |cb| \). These along with Lemma 4 imply that \( ad \) and \( db \) are the only candidates for a shortest edge of \( acbd \). Assume first that \( ad \) is a shortest edge of \( acbd \) (see Figure 4a).

Since \( \overrightarrow{ab} \in Y_4 \), \( |ab| \leq \sqrt{2}|ac| \). Since \( \overrightarrow{cd} \in Y_4 \), \( |cd| \leq \sqrt{2}|cb| \). These along with Lemma 4 imply that \( ad \) and \( db \) are the only candidates for a shortest edge of \( acbd \). Assume first that \( ad \) is a shortest edge of \( acbd \) (see Figure 4a).

Define

\[
\mathcal{P}_b = \mathcal{P}_R(b \to d) \oplus \mathcal{P}_R(y \to d) \\
\mathcal{P}_d = \mathcal{P}_R(d \to y)
\]

By Lemma 3, \( \mathcal{P}_R(y \to d) \) does not cross \( cd \). Then \( \mathcal{P}_b \) and \( \mathcal{P}_d \) must have a nonempty intersection. We now show that \( \mathcal{P}_b \) and \( \mathcal{P}_d \) satisfy conditions (i) and (iii) of the lemma. Proposition 1 applied on the quadrilateral \( xdyz \) tells us that \( |xc| + |yd| < |xy| + |cd| \). We also have that \( |cx| \geq |cd| \), since \( \overrightarrow{cd} \in Y_4 \) and \( x \) is in the same quadrant of \( c \) as \( d \). This along with the inequality above implies \( |yd| < |xy| \). Because \( xy \in \mathcal{P}_R(b \to d) \), by Lemma 2 we have that \( |xy| \leq |bd| \), which along with the previous inequality shows that \( |yd| < |bd| \).

This along with Lemma 2 shows that condition (iii) of the lemma is satisfied.
Furthermore, $|P_R(y \rightarrow d)| \leq |yd|\sqrt{2}$ and $|P_R(d \rightarrow y)| \leq |yd|\sqrt{2}$. It follows that $|P_a| + |P_d| \leq 3\sqrt{2}bd$.

3. $c \in Q_1(a)$, and $d \in Q_3(c)$, as in Figure 4b. Then $|ac| \geq \max\{ab, cd\}$, and by Lemma 4 ac is not a shortest edge of acbd. The case when bd is a shortest edge of acbd is settled by Lemmas 3 and 2: Lemma 3 tells us that $P_d = P_R(d \rightarrow b)$ does not cross ab, and $P_b = P_R(b \rightarrow d)$ does not cross cd. It follows that $P_d$ and $P_b$ have a nonempty intersection. Furthermore, Lemma 2 guarantees that $P_d$ and $P_b$ satisfy conditions (ii) and (iii) of the lemma. Consider now the case when ad is a shortest edge of acbd; the case when bc is shortest is symmetric. By Lemma 6, $P_R(d \rightarrow a)$ does not cross ab. If $P_R(a \rightarrow d)$ does not cross cd, then this case is settled: $P_d = P_R(d \rightarrow a)$ and $P_a = P_R(a \rightarrow d)$ satisfy the three conditions of the lemma. Otherwise, let $\overrightarrow{x'y} \in P_R(a \rightarrow d)$ be the edge crossing cd. Arguments similar to the ones used in case 1 above show that $P_a = P_R(a \rightarrow d) \oplus P_R(y \rightarrow d)$ and $P_d = P_R(d \rightarrow y)$ are two paths that satisfy the conditions of the lemma.

4. $c \in Q_1(a)$, and $d \in Q_4(c)$, as in Figure 4c. Note that a horizontal reflection of Figure 4c, followed by a rotation of $\pi/2$, depicts a case identical to case 1, which has already been settled.

5. $c \in Q_2(a)$, as in Figure 4d. Note that Figure 4d rotated by $\pi/2$ depicts a case identical to case 1, which has already been settled.

6. $c \in Q_3(a)$. Then it must be that $d \in Q_1(c)$, otherwise cd cannot cross ab. By Lemma 3 however, ab and cd may not cross, unless one of them is not in $Y_4$.

7. $c \in Q_4(a)$, as in Figure 4e. Note that a vertical reflection of Figure 4e depicts a case identical to case 1, so this case is settled as well.

We are now ready to establish the main lemma of this section, showing that there is a short path between the endpoints of two intersecting edges in $Y_4$.

**Lemma 8.** Let $a, b, c, d \in V$ be four distinct nodes such that $\overrightarrow{ab} \in Y_4$ crosses $\overrightarrow{cd} \in Y_4$, and let $xy$ be a shortest side of the quadrilateral abcd. Then $Y_4$ contains a path $p(x, y)$ connecting $x$ and $y$, of length $|p(x, y)| \leq \frac{\sqrt{2} - 1}{\sqrt{2} - 1} |xy|$. Furthermore, no edge on $p(x, y)$ is longer than $|xy|$.

**Proof.** Let $P_x$ and $P_y$ be the two paths whose existence in $Y_4$ is guaranteed by Lemma 7. By condition (iii) of Lemma 7, no edge on $P_x$ and $P_y$ is longer than $|xy|$. By condition (i) of Lemma 7, $P_x$ and $P_y$ have a nonempty intersection. If $P_x$ and $P_y$ share a node $u \in V$, then the path $p(x, y) = P_x[x, u] \oplus P_y[y, u]$ is a path from $x$ to $y$ in $Y_4$ no longer than $3\sqrt{2}|xy|$; the length restriction follows from guarantee (ii) of Lemma 7. Otherwise, let $d'b' \in P_x$ and $c'd' \in P_y$ be two edges crossing each other. Let $x'y'$ be a shortest side of the quadrilateral $a'b'c'd'$, with $x' \in P_x$ and $y' \in P_y$. Lemma 7 tells us that $|a'b'| \leq |xy|$ and $|c'd'| \leq |xy|$. These along with Lemma 4 imply that $|x'y'| \leq |xy|/\sqrt{2}$. This enables us to derive a recursive formula for computing a path $p(x, y) \in Y_4$ as follows:

$$p(x, y) = \begin{cases} x, & \text{if } x = y \\ P_x[x, x'] \oplus P_y[y, y'] \oplus p(x', y'), & \text{if } x \neq y \end{cases}$$

Simple induction on the length of $xy$ establishes the claim of the lemma. \qed
4 \( Y_4^\infty \) and \( Y_4 \)

We prove that every individual edge of \( Y_4^\infty \) is spanned by a short path in \( Y_4 \). This, along with the result of Theorem 1, establishes that \( Y_4 \) is a spanner. Fix an edge \( \overline{xy} \in Y_4^\infty \). Define an edge or a path as \( t\)-short (with respect to \( |xy| \)) if its length is within a constant factor \( t \) of \( |xy| \). In our proof that \( ab \) is spanned by a \( t \)-short path with respect to \( |ab| \) in \( Y_4 \), we will make use of the following three statements.

**S1** If \( ab \) is \( t \)-short, then \( \mathcal{P}_R(a \rightarrow b) \), and therefore its reverse, \( \mathcal{P}_R^{-1}(a \rightarrow b) \), are \( t\sqrt{2} \)-short by Lemma 2.

**S2** If \( ab \in Y_4 \) is \( t_1 \)-short and \( cd \in Y_4 \) is \( t_2 \)-short, and if \( ab \) intersects \( cd \), Lemmas 4 and 8 show that there is a \( t_3 \)-short path between any two of the endpoints of these edges with \( t_3 = t_1 + t_2 + 3(2 + \sqrt{2}) \max(t_1, t_2) \).

**S3** If \( p(a, b) \) is a \( t_1 \)-short path and \( p(c, d) \) is a \( t_2 \)-short path and the two paths intersect, then there is a \( t_3 \)-short path \( P \) between any two of the endpoints of these paths with \( t_3 = t_1 + t_2 + 3(2 + \sqrt{2}) \max(t_1, t_2) \), by **S2**.

**Lemma 9.** For any edge \( ab \in Y_4^\infty \), there is a path \( p(a, b) \in Y_4 \) between \( a \) and \( b \), of length \( |p(a, b)| \leq t|ab| \), for \( t = 29 + 23\sqrt{2} \).

**Proof.** For the sake of clarity, we only prove here that there is a short path \( p(a, b) \) between \( a \) and \( b \), and skip the calculations of the actual stretch factor \( t \) (which are detailed in the appendix of [1]). We refer to an edge or a path as short if its length is within a constant factor of \( |ab| \). Assume without loss of generality that \( \overline{ab} \in Y_4^\infty \), and \( \overline{ab} \in Q_1(a) \). If \( \overline{ab} \in Y_4 \), then \( p(a, b) = ab \) and the proof is finished. So assume the opposite, and let \( \overline{ac} \in Q_1(a) \) be the edge in \( Y_4 \); since \( Q_1(a) \) is nonempty, \( \overline{ac} \) exists. Because \( \overline{ac} \in Y_4 \) and \( b \) is in the same quadrant of \( a \) as \( c \), we have that

\[
|ac| \leq |ab| \quad (i)
\]

\[
|bc| \leq |ac|\sqrt{2} \quad (ii)
\]

Thus both \( ac \) and \( bc \) are short. And this in turn implies that \( \mathcal{P}_R(b \rightarrow c) \) is short by **S1**. We next focus on \( \mathcal{P}_R(b \rightarrow c) \). Let \( b’ \notin R(b, c) \) be the other endpoint of \( \mathcal{P}_R(b \rightarrow c) \). We distinguish three cases.

**Case 1:** \( \mathcal{P}_R(b \rightarrow c) \) and \( ac \) intersect. Then by **S3** there is a short path \( p(a, b) \) between \( a \) and \( b \).

**Case 2:** \( \mathcal{P}_R(b \rightarrow c) \) and \( ac \) do not intersect, and \( \mathcal{P}_R(b’ \rightarrow a) \) and \( ab \) do not intersect (see Figure 5b). Note that because \( b’ \) is the endpoint of the short path \( \mathcal{P}_R(b \rightarrow c) \), the triangle inequality on \( \triangle abb’ \) implies that \( ab’ \) is short, and therefore \( \mathcal{P}_R(b’ \rightarrow a) \) is short. We consider two cases:

(i) \( \mathcal{P}_R(b’ \rightarrow a) \) intersects \( ac \). Then by **S3** there is a short path \( p(a, b’) \). So

\[
p(a, b) = p(a, b’) \oplus \mathcal{P}_R^{-1}(b \rightarrow c)
\]

is short.
(ii) $P_R(b' \rightarrow a)$ does not intersect $ac$. Then $P_R(c \rightarrow b')$ must intersect $P_R(b \rightarrow c) \oplus P_R(b' \rightarrow a)$. Next we establish that $b'c$ is short. Let $eb'$ be the last edge of $P_R(b \rightarrow c)$, and so incident to $b'$ (note that $e$ and $b$ may coincide). Because $P_R(b \rightarrow c)$ does not intersect $ac$, $b'$ and $c$ are in the same quadrant for $e$. It follows that $|eb'| \leq |ec|$ and $\angle b'ec < \pi/2$. These along with Proposition 2 for $\triangle b'ec$ imply that $|b'c|^2 < |b'e|^2 + |ec|^2 \leq 2|ec|^2 < 2|bc|^2$ (this latter inequality uses the fact that $\angle bc > \pi/2$, which implies that $|ec| < |bc|$). It follows that

$$|b'c| \leq |bc|\sqrt{2} \leq 2|ac| \quad \text{(by (8)ii)} \quad (9)$$

Thus $b'c$ is short, and by S1 we have that $P_R(c \rightarrow b')$ is short. Since $P_R(c \rightarrow b')$ intersects the short path $P_R(b \rightarrow c) \oplus P_R(b' \rightarrow a)$, there is by S3 a short path $p(c, b)$, and so

$$p(a, b) = ac \oplus p(c, b)$$

is short.

**Case 3:** $P_R(b \rightarrow c)$ and $ac$ do not intersect, and $P_R(b' \rightarrow a)$ intersects $ab$ (see Figure 5c). If $P_R(b' \rightarrow a)$ intersects $ab$ at $a$, then $p(a, b) = P_R(b \rightarrow c) \oplus P_R(b' \rightarrow a)$ is short. So assume otherwise, in which case there is an edge $de \in P_R(b' \rightarrow a)$ that crosses $ab$. Then $d \in Q_4(a)$, $e \in Q_4(a) \cup Q_4(a)$, and $e$ and $a$ are in the same quadrant for $d$. Note however that $e$ cannot lie in $Q_3(a)$, since in that case $\angle ade > \pi/2$, which would imply $|de| > |da|$, which in turn would imply $de \not\in Y_4$. So it must be that $e \in Q_4(a)$.

Next we show that $P_R(e \rightarrow a)$ does not cross $ab$. Assume the opposite, and let $rs \in P_R(e \rightarrow a)$ cross $ab$. Then $r \in Q_4(a)$, $s \in Q_4(a) \cup Q_2(a)$, and $s$ and $a$ are in the same quadrant for $r$. Arguments similar to the ones above show that $s \not\in Q_3(a)$, so $s$ must lie in $Q_1(a)$. Let $d$ be the $L_\infty$ distance from $a$ to $b$. Let $x$ be the projection of $r$ on the horizontal line through $a$. Then

$$|rs| \geq |rx| + d \geq |rx| + |xa| > |ra| \quad \text{(by the triangle inequality)}$$

**Fig. 5.** Lemma 9: (a) Case 1: $P_R(b \rightarrow c)$ and $ac$ have a nonempty intersection. (b) Case 2: $P_R(b' \rightarrow a)$ and $ab$ have an empty intersection. (c) Case 3: $P_R(b' \rightarrow a)$ and $ab$ have a non-empty intersection.
Because $a$ and $s$ are in the same quadrant for $r$, the inequality above contradicts $rs \in Y_4$.

We have established that $P_R(e \rightarrow a)$ does not cross $ab$. Then $P_R(a \rightarrow e)$ must intersect $P_R(e \rightarrow a) \oplus de$. Note that $de$ is short because it is in the short path $P_R(b' \rightarrow a)$. Thus $ae$ is short, and so $P_R(a \rightarrow e)$ and $P_R(e \rightarrow a)$ are short. Thus we have two intersecting short paths, and so by $S3$ there is a short path $p(a,e)$. Then

$$p(a,b) = p(a,e) \oplus P_R^{-1}(b' \rightarrow a) \oplus P_R^{-1}(b \rightarrow c)$$

is short. Straightforward calculations show that, in each of these cases, the stretch factor for $p(a,b)$ does not exceed $29 + 23\sqrt{2}$. \hfill \Box

Our main result follows immediately from Theorem 1 and Lemma 9:

**Theorem 2.** $Y_4$ is a $t$-spanner, for $t \geq 8\sqrt{2}(29 + 23\sqrt{2})$.

## 5 Conclusion

Our results settle a long-standing open problem, asking whether $Y_4$ is a spanner or not. We answer this question positively, and establish a loose stretch factor of $8\sqrt{2}(29 + 23\sqrt{2})$. Experimental results, however, indicate a stretch factor of the order $1 + \sqrt{2}$, a factor of 200 smaller. Finding tighter stretch factors for both $Y_4^\infty$ and $Y_4$ remain interesting open problems. Establishing whether $Y_5$ and $Y_6$ are spanners or not is also open.

### References