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Expressing Vague Knowledge in the Fuzzy Description Logic

fALCHIN

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Abstract

Uncertainty is an intrinsic feature of our knowledge, which is also reflected by the World Wide World and the Semantic Web. Motivated by Web applications, this paper introduces an expressive fuzzy description logic that extends classical description logics to many-valued logics. The syntax to represent imprecise or vague knowledge and the semantics to interpret complex concept descriptions are addressed in detail. This proposed fuzzy description logic, $fALCHIN$, extends the expressiveness of the well know ALC description logic by fuzzy inverse role and fuzzy role inclusion axiom, as well as fuzzy at-most/at least number restrictions.

1 Introduction

The Semantic Web initiative aims at creating an extension of the current World Wide Web by developing logic-based standards and technologies that enable machines to understand the information on the web, so that they can support richer knowledge discovery and automate the performance of various tasks for human beings[1].

An key research direction for the Semantic Web is to handle uncertainty, as evidenced by Fuzzy RuleML [2] and W3C's Uncertainty Reasoning for the World Wide Web Incubator Group [3]. Typical Description Logics (DL) are limited to dealing with crisp, well defined concepts. They cannot express vague or uncertain knowledge. However, uncertainty is an intrinsic feature of real-world knowledge. Many concepts needed in knowledge modeling lack well-defined boundaries or, precisely defined criteria of relationships with other concepts. For example, the concept of young man, tall, and cold.

To overcome this deficiency, this paper proposed an extension to Description Logics based on Fuzzy Logic. The rest of this paper is organized as follows. Section 2 briefly introduces the syntax and semantics of expressive Description Logics. Section 3 reviews the Fuzzy Logic and Fuzzy theory. Section 4 presents the syntax and semantics of an expressive fuzzy description logic, as well as the components of a knowledge base using such this knowledge representation formalism. Section 5 reviews some related work in uncertainty management in Description Logic. Finally, in Section 6 we summarize our main results and give an outlook on future research.

2 Preliminaries

We briefly introduce Description Logics in the current section. Their syntax and semantics in terms of classical First Order Logic are also presented. As a notational convention, we will use a,b,x for individuals, A for atomic concepts, C and D for concept descriptions, R and P for atomic roles.

Description Logics (DL) [4, 5] are a family of logic-based knowledge representation formalisms designed to represent and reason about the knowledge of a concrete domain. Elementary descriptions of DL are atomic concepts and atomic roles. Complex concept descriptions can be built from the elementary constructors and constructing rules. Different description languages of DL are distinguished by the constructors they provide. For example, $ALCHIN$ DL extends the well known ALC DL with inverse roles, role inclusion axioms, and number restrictions. Concept constructors in $ALCHIN$ are formed according to the syntaxes in Table 1.

Table 1. Syntax and Semantics of $SHIN$ constructors

DL Constructor	DL Syntax	Semantics
top concept	T	Δ^I
bottom concept	$\bot$	$\varnothing$
atomic concept	A	$A^I \subseteq \Delta^I$
concept name	C	$C^I \subseteq \Delta^I$
atomic negation	$\neg A$	$\Delta^I \setminus A^I$
concept negation	$\neg C$	$\Delta^I \setminus C^I$
concept conjunction	$C \sqcap D$	$C^I \bigcap D^I$
concept disjunction	$C \sqcup D$	$C^I \bigcup D^I$
exists restriction	$\exists R.C$	$\{x \in \Delta^I \mid \exists y. < x, y > \in R^I \wedge y \in C^I\}$
value restriction	$\forall R.C$	$\{x \in \Delta^I \mid \forall y. < x, y > \in R^I \rightarrow y \in C^I\}$
inverse role	R^-	$(R^-)^I(y,x) = R^I(x,y)$
at-most restriction	$\leq n\ R$	$\{x \in \Delta^I \mid \#\{y \in \Delta^I \mid R^I(x,y)\} \leq n\}$
at-least restriction	$\geq n\ R$	$\{x \in \Delta^I \mid \#\{y \in \Delta^I \mid R^I(x,y)\} \geq n\}$

Description Logics have a model theoretic semantics, which is defined by interpreting concepts as sets of individuals and roles as sets of pairs of individuals. An interpretation I is a pair $I = (\Delta^I, \cdot^I)$ consisting of a domain Δ^I which is a non empty set and of an interpretation function $\cdot^I$ which maps each individual x into an element of Δ^I ($x^I \in \Delta^I$), each concept C into a subset of Δ^I ($C^I \subseteq \Delta^I$) and each atomic role R into a subset of $\Delta^I \times \Delta^I$ ($R^I \subseteq \Delta^I \times \Delta^I$). The interpretations of complex concept descriptions are shown in Table 1.

A knowledge base (KB) based on DL $KB = <T, A>$ consists of two parts: the terminological box (TBox T) and the assertion box (ABox A). There are two kinds of assertions in the ABox of a DL KB: concept individual and role individual. A concept instance assertion has the form $C(a)$ while a role instance assertion is $R(a,b)$. The semantics of assertions is interpreted as the assertion $C(a)$ (resp. $R(a,b)$) is satisfied by I iff $a^I \in C^I$ (resp. $(a^I, b^I) \in R^I$).

A DL KB has several kinds of axioms. A concept inclusion axiom is an expression of subsumption with the form $C \sqsubseteq D$. The semantics of a concept inclusion axiom is interpreted as the axiom is satisfied by I iff $\left\{x \in \Delta^I \mid \forall x, x \in C^I \rightarrow x \in D^I\right\}$. A concept equivalence axiom is an expression of the form $C \equiv D$. Its semantics is that the axiom is satisfied by I iff $\left\{x \in \Delta^I \mid \forall x, x \in C^I \rightarrow x \in D^I, x \in D^I \rightarrow x \in C^I\right\}$. An inverse role axiom is of the form $R^- \equiv R$ with the semantics interpreted as the axiom is satisfied by I iff $\left\{x, y \in \Delta^I \mid (R^-)^I(y,x) = R^I(x,y)\right\}$. An role inclusion axiom has the form $R \sqsubseteq P$ with its semantic states that the axiom is satisfied by I iff $\left\{x, y \in \Delta^I \mid R^I(x,y) \rightarrow P^I(x,y)\right\}$. Similarly, we can define the syntax of a role equivalence axiom as $R \equiv P$ and its semantics.

3 Fuzzy Set Theory and Fuzzy Logic

Fuzzy set theory was first introduced by Zadeh [6] as an extension of the classical notion of set to capture the inherent vagueness (the lack of crisp boundaries of sets). Fuzzy logic is a form of multi-valued logic derived from fuzzy set theory to deal with reasoning that is approximate rather than precise. Just as in fuzzy set theory the set membership values can range between 0 and 1, in fuzzy logic the degree of truth of a statement can range between 0 and 1 and is not constrained to the two truth values {true, false} as in classic predicate logic[7]. Formally, a fuzzy set X with respect to a set of elements Ω (also called a universe) is characterized by a membership function $\mu(x)$ which assigns a value in the real unit interval [0,1] to each element x in X $(x \in X)$. $\mu(x)$ gives us an estimation that an element x belongs to a set A to a certain degree. Such degrees could be computed based on some specific membership functions. Fig. 1 summarizes the most frequently used crisp, trapezoidal, triangular, left-shoulder, and right-shoulder membership functions. Here we define these functions as $crisp(a,b)$, $leftshoulder(a,b)$, $rightshoulder(a,b)$, $triangular(a,b,c)$, and $trapezoidal(a,b,c,d)$ respectively. The domain of these membership functions are defined as $[k_1, k_2]$.

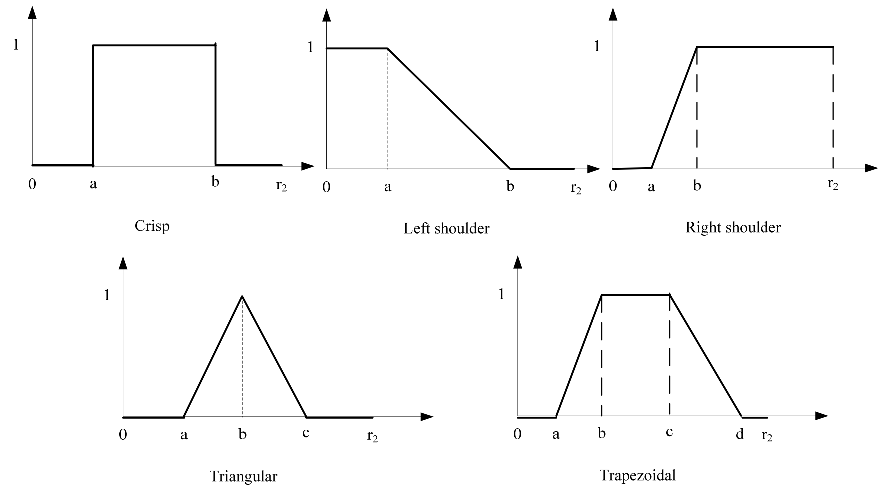

Fig. 1 Fuzzy Membership Functions

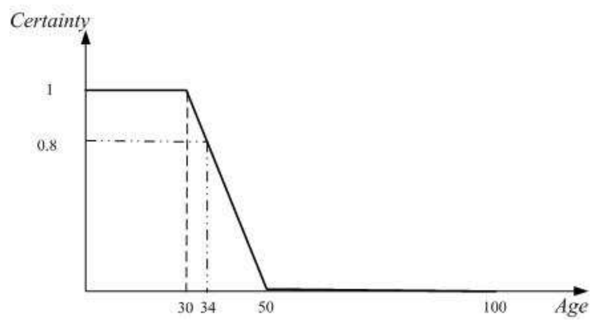

Fig. 2 A left_shoulder membership function for the concept Young

For example, a fuzzy set $Young$ is defined by a left_shoulder membership function $leftshoulder(30,50)$ as shown in Fig. 2. Now we know, John is 34 years old. Therefore, we have $Young(John) = 0.8$ which means the statement "John is a young man" has a truth value of 0.8. But more often, we want to make vaguer statements, saying that "John is a young man" has a truth value of greater than or equal to 0.8. Such a statement can be written as $Young(John) \geq 0.8$. Another kind of mainly used statement is less than or equal to. In order to describe all the above statements in a unified form, we propose a syntax as $[l,u]$ $(0 \leq l \leq u \leq 1)$. Therefore, $Young(John) \geq 0.8$ can be written as $Young(John) = [0.8,1]$, $Young(John) = 0.8$ as $Young(John) = [0.8,0.8]$, and $Young(John) \leq 0.8$ as $Young(John) = [0,0.8]$.

A fuzzy relation R is over two fuzzy sets X_1 and X_2 is defined by a function $R : \Omega \times \Omega \rightarrow [0,1]$. For example, the statement "Young people drive fast" has a truth value of greater than or equal to 0.6 can be defined as a fuzzy relation R over two fuzzy sets *Young* and *Fast* : $R(John, 150) = [0.6, 1]$.

Fuzzy logic extends the Boolean operations such as complement, union, and intersection, defined on crisp sets and relations in the context of fuzzy sets and fuzzy relations. These operations are interpreted as mathematical functions over the unit interval [0,1]. The mathematical functions for fuzzy intersection are usually called t-norms, those for fuzzy union are called s-norms, and the fuzzy set complement is called negation. Different types of such operations in Fuzzy Logic including Zadeh Logic, Lukasiewicz Logic, Product Logic, and Gödel Logic, are summarized in Table 2. All these operations satisfy certain mathematical properties.

Table 2 Fuzzy Opertaions

	Zadeh	Lukasiewicz Logic	Product Logic	Gödel Logic
t-norms($t(x,y)$)	$\min(x,y)$	$\max(x+y-1,0)$	$x \cdot y$	$\min(x,y)$
s-norms($s(x,y)$)	$\max(x,y)$	$\min(x+y,1)$	$x+y-x \cdot y$	$\max(x,y)$
negation($\neg x$)	$1-x$	$1-x$	$if\ x=0\ then\ 1\ else\ 0$	$if\ x=0\ then\ 1\ else\ 0$

4 Fuzzy Description Logic

4.1 Syntax of *fALCHIN*

Concept descriptions in *fALCHIN* are formed based on the following syntax:

$$C \to \top \mid \bot \mid A \mid C \sqcap D \mid C \sqcup D \mid \exists R.C \mid \forall R.C \mid \geq nR \mid \leq nR$$

We can see that the syntax of this fuzzy description logic is identical to that of the standard description logics. But here in *fALCHIN*, the concepts and roles are defined as fuzzy concepts (i.e. fuzzy sets) and fuzzy roles (i.e. fuzzy relations).

4.2 Semantics of *fALCHIN*

Similar to classical DL, the semantics of the proposed *fALCHIN* is based on the notion of interpretation. Classical interpretations is extended to the concept of fuzzy interpretations by using membership functions that range over the interval [0,1]. An fuzzy interpretation I is a still pair $I = (\Delta^I, \cdot^I)$ consisting of a domain Δ^I which is a non empty set and of a fuzzy interpretation function $\cdot^I$ which maps each individual x into an element of Δ^I ($x^I \in \Delta^I$), each concept C into a membership function of $C^I : \Delta^I \to [0,1]$, and each atomic role R into a membership function of $R^I : \Delta^I \times \Delta^I \to [0,1]$.

Next we define the semantics of *fALCHIN* constructors, including the top concept, the bottom concept, concept negation, concept conjunction, concept disjunction, role exists restriction, role value restriction, and number restrictions. We explain how to apply the fuzzy logic operations in Table 2 to the proposed *fALCHIN* with some examples.

The semantics of the top concept T is the greatest element in the domain Δ^I, that is, $T^I(x) = 1(\forall x, x \in \Delta^I)$. Please note that, in classical DL, the top concept $T \equiv A \sqcup \neg A$, while in $f-SHIN$, $T \neq A \sqcup \neg A$. As shown in Table 2, after applying the s-norms on $A \sqcup \neg A$, the result is no longer 1, which is contradictory to intuition. Therefore, we explicitly define the top concept, stating that the truth degree of x in T is 1. Similarly, the bottom concept $\perp$ is the least element in the domain, defined as $\perp^I(x) = 0(\forall x, x \in \Delta^I)$.

The concept negation (also known as concept complement) $\neg C$ is interpreted with a mathematical function which satisfies

a). $\neg T^I(x) = 0, \neg \perp^I(x) = 1$.

b). self-inverse, i.e., $(\neg\neg C)^I(x) = C^I(x)$.

For example, if we have the statement "John is a young man" has a truth value of greater than or equal to 0.8 ($Young(John) = [0.8,1]$), and assume we choose the negation operator in Zadeh logic or Lukasiewicz logic, then the statement "John is not a young man" is written as $\neg Young(John) = \neg[0.8,1] = [0,0.2]$.

The interpretation of concept conjunction (also called concept intersection) is defined by t-norms as

$$(C \sqcap D)^I(x) = t(C^I(x), D^I(x))\ (\forall x, x \in \Delta^I)$$

For example, if we have $Young(John) = [0.8,1]$ and $Tall(John) = [0.7,1]$, and assume the minimum function is chosen as the t-norm, then the certainty that John is both young and tall is $(Young \sqcap Tall)(John) = \min([0.8,1],[0.7,1]) = [0.7,1]$.

The interpretation of concept disjunction/union is defined by the s-norms as

$$(C \sqcup D)^I(x) = s(C^I(x), D^I(x))\ (\forall x, x \in \Delta^I)$$

For example, if we have $Young(John) = [0.8,1]$ and $Tall(John) = [0.7,1]$, and the s-norm is maximum, then the certainty that John is either young or tall is $(Young \sqcup Tall)(John) = \min([0.8,1],[0.7,1]) = [0.8,1]$.

The semantics of role exists restriction $\exists R.C$ is the result of viewing $\exists R.C$ as the open first order formula $\exists y.F_R(x,y)\wedge F_C(y)$ and the existential quantifier $\exists$ is viewed as a disjunction over the elements of the domain. Therefore, we define

$$(\exists R.C)^I(x)=\sup_{y\in\Delta^I}\{t(R^I(x,y),C^I(y))\}$$

Suppose we have $hasFatalDisease(John,Cancer)=[0.2,1]$, $FatalDisease(Cancer)=[0.5,1]$, $hasFatalDisease(John,Cold)=[0.6,1]$, and $FatalDisease(Cold)=[0.1,1]$. Further we assume the minimum function is chosen as the t-norm, then

$$\begin{aligned}(\exists R.C)^I(x)=\sup\{&\min(hasFatalDisease(John,Cancer),FatalDisease(Cancer)),\\ &\min(hasFatalDisease(John,Cold),FatalDisease(Cold))\}\\ =&\sup\{\min([0.2,1],[0.5,1]),\min([0.6,1],[0.1,1])\}\\ =&\sup\{[0.2,1],[0.1,1]\}=[0.2,1]\end{aligned}$$

That is, the truth degree for the complex concept assertion $(\exists hasFatalDisease.FatalDisease)(John)$ is greater than or equal to 0.2.

A role value restriction $\forall R.C$ is viewed as an implication of the form $\forall y\in\Delta^I, R^I(x,y)\rightarrow C^I(x)$. As proposed by Hajek [8], we interpret $\forall$ as inf. Furthermore, in classical logic, $a\rightarrow b$ is a shortand for $\neg a\vee b$, we can thus interpret $\rightarrow$ as the Kleene-Dienes implication and finally get its semantics as $(\forall R.C)^I(x)=\inf_{y\in\Delta^I}\{s(\neg R^I(x,y),C^I(y))\}$.

A fuzzy at-least restriction is of the form $\geq n\,R$ whose semantic $(\geq nR)^I(x)=\sup_{\substack{y_1,\cdots,y_n\in\Delta^I\\ y_i\neq y_j,1\leq i<j\leq n}} t_{i=1}^{n}\{R^I(x,y_i)\}$ is derived from its first order formula

$$\exists y_1,\cdots,y_n.\wedge_{i=1}^{n}R(x,y_i)\wedge\bigwedge_{1\leq i<j\leq n}y_i\neq y_j.$$

The semantics states that there are at least n distinct elements that satisfy to some degree.

Furthermore, since $\leq nR\equiv\neg(\geq(n+1)R)$, we define the semantics of a fuzzy at-most restriction as

$$\begin{aligned}(\leq nR)^I(x)&=\neg(\geq(n+1)R)^I(x)=\neg(\sup_{\substack{y_1,\cdots,y_{n+1}\in\Delta^I\\ y_i\neq y_j,1\leq i<j\leq n+1}} t_{i=1}^{n+1}\{R^I(x,y_i)\})\\ &=\inf_{\substack{y_1,\cdots,y_{n+1}\in\Delta^I\\ y_i\neq y_j,1\leq i<j\leq n+1}} s_{i=1}^{n+1}\{R^I(x,y_i)\}\end{aligned}.$$

The FOL translation of a concept inclusion axiom $C\sqsubseteq D$ has the form

$\forall x.C(x) \to D(x)$, therefore, its semantics is defined as

$$(C \sqsubseteq D)^I(x) = \inf_{x \in \Delta^I} C^I(x) \to D^I(x) = \inf_{x \in \Delta^I} \{s(\neg C^I(x), D^I(x))\}.$$

Similarly, the semantics of a role inclusion axiom $R \sqsubseteq P$ is

$$(R \sqsubseteq P)^I(x, y) = \inf_{x, y \in \Delta^I} \{s(\neg R^I(x, y), P^I(x, y))\}.$$

The semantics of the complex concept descriptions for *fALCHIN* are summarized in Table 3.

Table 3. Syntax and Semantics of *fALCHIN* DL constructors, axioms and assertions

DL Constructor	DL Syntax	Semantics
top concept	T	$T^I(x) = 1$
bottom concept	$\perp$	$\perp^I(x) = 0$
concept negation	$\neg C$	$(\neg C)^I(x) = \neg C^I(x)$
concept conjunction	$C \sqcap D$	$(C \sqcap D)^I(x) = t(C^I(x), D^I(x))$
concept disjunction	$C \sqcup D$	$(C \sqcup D)^I(x) = s(C^I(x), D^I(x))$
exists restriction	$\exists R.C$	$(\exists R.C)^I(x) = \sup_{y \in \Delta^I} \{t(R^I(x, y), C^I(y))\}$
value restriction	$\forall R.C$	$(\forall R.C)^I(x) = \inf_{y \in \Delta^I} \{s(\neg R^I(x, y), C^I(y))\}$
at-least restriction	$\geq n\ R$	$(\geq nR)^I(x) = \sup_{\substack{y_1, \cdots, y_n \in \Delta^I \\ y_i \neq y_j, 1 \leq i < j \leq n}} t_{i=1}^n \{R^I(x, y_i)\}$
at-most restriction	$\leq n\ R$	$(\leq nR)^I(x) = \neg(\geq (n+1)R)^I(x)$
inverse role	R^-	$(R^-)^I(y, x) = R^I(x, y)$
concept inclusion axiom	$C \sqsubseteq D$	$(C \sqsubseteq D)^I(x) = \inf_{x \in \Delta^I} \{s(\neg C^I(x), D^I(x))\}$
role inclusion axiom	$R \sqsubseteq P$	$(R \sqsubseteq P)^I(x, y) = \inf_{x, y \in \Delta^I} \{s(\neg R^I(x, y), P^I(x, y))\}$
concept instance assertion	$C(a)$	$C^I(a^I)$
role instance assertion	$R(a,b)$	$R^I(a^I, b^I)$

4.3 Knowledge Base in *fALCHIN*

A fuzzy knowledge base in *fALCHIN* consists of a finite set of fuzzy axioms and fuzzy assertions. A fuzzy concept inclusion axiom has a form of $C \sqsubseteq D\,[l,u]$ $(0 \le l \le u \le 1)$ which describes that the subsumption degree between concept C and D is from l to u.

For example, the axiom

Professor $\sqsubseteq$ ($\exists$publishes.Journalpaper $\sqcap$ $\exists$teaches.Graduatecourse) [0.8,1]

states that the concept professor is subsumed by publishing journal papers and teaching graduate courses with a certainty degree of at least 0.8.

A fuzzy role inclusion axiom has the form $R \sqsubseteq P\,[l,u]$. A fuzzy concept assertion and a fuzzy role assertion are of the form $C(a)\,[l,u]$ and the form $R(a,b)\,[l,u]$ respectively.

5 Related Work

Uncertainty is known as an intrinsic feature of the World Wide World and Semantic Web. W3C even founded a group, the Uncertainty Reasoning for the World Wide Web (URW3) Incubator Group, which is dedicated to define the challenge of representing and reasoning with uncertain information. According to the latest URW3 draft report, uncertainty is a term intended to include different forms of incomplete knowledge, including incompleteness, inconclusiveness, vagueness, ambiguity, and others[3]. Mathematical theories for representing uncertainty information includes, but not limited to, probability, Fuzzy Sets, Belief Functions, Random Sets, Rough Sets, and combination of several models (Hybrid).

There has been some work carried out in integrating uncertainty knowledge into Description Logics in the last decades [9-15]. Current literature generally can be divided into two approaches. One is based on probabilistic theory [9, 10, 15] and the other is based on fuzzy logic [11-14, 16]. Although both approaches assign numerical values to entries in a knowledge base, they are quite different; not only from a technical point of view, but also with respect to the basic phenomena they are trying to model. Probabilistic theory refers to a proposition that is either true or false, but due to a lack of information we do not know for certain which one is the case. It represents the probability with which a proposition is assumed to be true. For example, John can be assumed to be a student with the probability 0.6 and a teacher with the probability 0.4. On the other hand, fuzzy logic is used to represent the vagueness of a proposition, which means the proposition itself is only true to a certain degree. For example, John, measuring 1.85m, might be said to be tall with the degree of truth 0.9.

Our fuzzy description logic extended the expressiveness of the fuzzy ALC in [11, 16] to support the at-least and at-most number restriction constructors, as well as the inverse and transitive role. Unlike other approaches based on fuzzy logic [11, 12, 14, 16, 17] which only deal with crisp subsumption of fuzzy concepts, our fuzzy description logic deals with fuzzy subsumptions of fuzzy concepts and addresses its semantics. We believe that fuzzy subsumption of fuzzy concepts in the form of $C \sqsubseteq D$ $[l,u]$ is closer to the uncertain knowledge existing in the real world applications. [13] first proposed the notion of fuzzy subsumption but only used a form $\geq n$, while our approach generalizes it to a range of certainty values.

6 Conclusion and Future Work

In this paper, we proposed an extension to Description Logics based on Fuzzy Set Theory and Fuzzy Logic. The syntax and semantics of the proposed description logic $fALCHIN$ were explained in details. We also addressed the components of a $fALCHIN$ knowledge base.

Description Logics is a family of description languages with different expressiveness. Our fuzzy description language extends the fuzzy ALC and takes into account inverse roles, role inclusion axioms, and number restrictions, but leaves alone transitive roles, nominals (i.e. collection of individuals) and datatypes for the reason of simplicity. Future work will include a fuzzy extension to more expressive description languages.

From the point view of reasoning with a $fALCHIN$ knowledge base, we present different reasoning tasks and the reasoning algorithm in another upcoming paper, because of the length limit here.

From the point view of implementing a corresponding reasoner, the plan is to build it on top of Pellet (http://clarkparsia.com/pellet/). Pellet is an open-source Java based OWL DL reasoner. Our extension of Pellet will provide functionalities to check consistency, entailments and subsumptions of a knowledgebase.

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