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Path integral formulation of optical beam propagation

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Propagation of optical beams described by the Schrödinger type equations is expressed in terms of the propagator obtained in the framework of the Feynman path integral formulation of quantum mechanics. A numerical scheme is developed to evaluate the propagator by the fast Fourier transform scheme. The resulting procedure is more convenient, efficient, versatile and accurate than the alternative methods in use. The method is illustrated with several examples including the soliton propagation and the converging beams in a self-focusing medium together with the impact of plasma it generates.

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1. Introduction

Propagation properties of the optical beams are required in the studies of a number of physical phenomena. Understanding of the behavior of an optical beam propagating in a dielectric medium is essential to describe the laser-material interactions and other related processes¹⁻⁷. Amplitude of an optical beam in general satisfies a nonlinear Schrödinger type equation, and thus, the considerations are applicable to the quantum mechanical problems also⁸. Propagation of an optical beam in a nonlinear photonic crystal is described by a coupled set of nonlinear Schrödinger type equations in the slowly varying envelope approximation.⁹⁻¹³ Basic properties of such equations are essentially the same as those of the uncoupled nonlinear and linear equations encountered in the other areas. Thus, the interest in an analysis and the solution methods for the Schrödinger type equations is of a wide scope.

For some simple cases, the solutions of the Schrödinger type equations can be obtained analytically¹³ but for realistic situations, it is necessary to resort to semi-analytical^{11,12,14} or completely numerical procedures.¹⁻⁸ These schemes are based on the variational methods^{11,12,14}, finite difference schemes¹⁻⁴ and the fast Fourier transforms (*FFT*)⁵⁻⁸. Semi-analytical methods are quite cumbersome and the finite difference schemes accumulate significant round offs. Application of *FFT* is quite efficient as well as convenient and due to the availability of reliable routines, produces commensurate results. For these reasons, the procedure has been used frequently. However, the method has some limitations creating difficulties in its numerical applications.⁷ Further, this and other methods experience difficulties when applied to the case of a rapidly converging or diverging beam. For example, in case of a focused beam, the beam radius of the order of millimeters at the

lens may converge on to the radius of the order of micrometers at the focal point. The resulting difficulty is overcome at times by questionable adjustments. For a proper resolution, a variable mesh structure and unequal mesh size is more suitable, which requires cumbersome adjustments to the original procedures.¹⁴ While a variety of methods have been successfully employed to study the properties of the optical beams in diverse situations, further improvements and developments are still desirable.

Time-evolution of the solutions of the linear Schrödinger equation can be expressed in terms of a propagator in the framework of the path integral formulation of quantum mechanics.¹⁵ This description can be extended to include the nonlinear and coupled sets of Schrödinger equations in a straightforward manner, as well as to other differential equations. While this formulation has proved to be quite useful in treating a variety of topics in physics, to the author's best knowledge, it has not been exploited for numerical evaluations of the solutions. In the present article, a numerical scheme is developed to determine the evolution of the solutions of the Schrödinger equations based on their path-integral propagator representations. It is shown that this procedure can be conveniently implemented by the *FFT* schemes yielding the solution in about half the number of operations required to implement the other methods based on *FFT*, rendering it even more efficient. Since the number of numerical operations is decreased considerably, it is also more accurate and since the *FFT* routines are quite well developed and widely used, it is reliable as well. Further, the propagation of the beams with rapidly variable beam radius is re-formulated eliminating the complications associated with the converging and diverging beams thereby rendering it suitable for the application of a number of methods including the present. The results

obtained are illustrated with the propagation of solitons and solitary waves encountered in the photonic band gap crystals and other mediums as well as with rapidly converging beams in self-focusing dielectric mediums together with the distortions caused by the plasma generated. In all of the cases the present method was found to be straightforward to implement, very efficient and accurate.

2. Formulation

Consider the Schrodinger equation

$$i \frac{\partial u}{\partial \tau} + \frac{1}{2} \nabla^2 u - V u = 0, \quad (1)$$

where $V(x)$ is the potential. Path-integral propagator representation of the solutions of the one dimensional case of Eq. (1) accurate up to $o(\varepsilon)$ is given by¹⁵

$$u(x, \tau + \varepsilon) = \frac{1}{\sqrt{2\pi i \varepsilon}} \int_{-\infty}^{\infty} dx' \exp \left[\frac{i}{2\varepsilon} (x - x')^2 \right] \exp[-i\varepsilon V(x, \tau)] u(x', \tau). \quad (2)$$

The free particle propagator is exactly determined by taking $V = 0$:

$$\begin{aligned} (P(\varepsilon)h(\tau))(x, \tau + \varepsilon) &= \left(\exp[i\varepsilon \nabla^2] h(\tau) \right)(x, \tau + \varepsilon) \\ &= \frac{1}{\sqrt{2\pi i \varepsilon}} \int_{-\infty}^{\infty} dx' \exp \left[\frac{i}{2\varepsilon} (x - x')^2 \right] h(x', \tau). \end{aligned} \quad (3)$$

The terms can be arranged to determine the solution accurate up to $o(\varepsilon^2)$ yielding^{7,16}

$$u(\tau + 2\varepsilon) = P(\varepsilon) \exp[-2i\varepsilon V(\tau)] P(\varepsilon) u(\tau), \quad (4)$$

where the space variables are suppressed for convenience. Eqs. (2) and (4) are extended for a solution dependent $V(u, x, \tau)$ by direct substitution, which covers the type of nonlinear cases of interest. Extension to the multi-dimensional cases poses no analytical difficulties. The results are valid with $P(\varepsilon)$ replaced with the multidimensional free propagator, which is the product of one-dimensional propagators, each one essentially identical to $P(\varepsilon)$.

The equation

$$i \frac{\partial u'}{\partial \tau} + \frac{1}{2} \alpha \nabla^2 u' - V u' = 0, \quad (5)$$

can be treated by the same method as Eq. (1) to deduce that

$$u'(\tau + 2\varepsilon) = P(\alpha \varepsilon) \exp[-2i\varepsilon V(\tau)] P(\alpha \varepsilon) u'(\tau), \quad (6)$$

for a real α . The procedure also extends to treat the coupled sets of equations,

$$i \frac{\partial \nu}{\partial \tau} + L \nu - V \nu = 0, \quad (7)$$

where the solution ν is a $\mathcal{N}$ -dimensional vector and V is a $\mathcal{N} \times \mathcal{N}$ matrix with functions as their elements. It will be assumed that L is a diagonal matrix with elements $(\alpha_\eta \nabla^2 / 2)$, $\eta = 1, 2, \dots, \mathcal{N}$. It follows from Eqs. (4) and (6) that

$$\nu(\tau + 2\varepsilon) = Q(\varepsilon) \exp[-2i\varepsilon V(\tau)] Q(\varepsilon) \nu(\tau), \quad (8)$$

where $Q(\varepsilon)$ is a diagonal matrix with $Q_{\eta\eta}(\varepsilon) = P(\alpha_\eta \varepsilon)$, $\eta=1,2,\dots, \mathcal{N}$. Exponentiation of the matrix V can be avoided by replacing it with $(1-2i\varepsilon V+2\varepsilon^2 V^2)$, which still maintains the same order of accuracy. Thus the formulation covers a large class of cases.

Propagation of an optical beam in a dielectric medium is described by the following non-linear Schrödinger equation^{1-7,17}

$$i \frac{\partial A}{\partial z'} = - \frac{1}{2k} \nabla_{\mathbf{r}'}^2 A + \hat{V}(I) A, \quad (9)$$

satisfied by slowly varying amplitude function A , with intensity $I = |A|^2$. Here $\nabla_{\mathbf{r}'}^2$ denotes the Laplacian in the plane $\mathbf{r}' = (x', y')$ perpendicular to the axis of the beam z' , and $k = (2\pi/\lambda)$ is the wave number with λ being the wavelength. For convenience, Eq. (9) is expressed in dimensionless form

$$i \frac{\partial A}{\partial z} = - \frac{1}{2} \nabla_{\mathbf{r}'}^2 A + w_0 \hat{V}(I) A, \quad (10)$$

where $\mathbf{r} = \mathbf{r}' \sqrt{k/w_0}$ and $z = z'/w_0$ with w_0 being the beam radius at a convenient location.

For propagation in vacuum, the perturbation term drops out reducing Eq. (10) to the Helmholtz equation

$$i \frac{\partial A_0}{\partial z} = - \frac{1}{2} \nabla_{\mathbf{r}'}^2 A_0. \quad (11)$$

Solutions of Eq. (15) of practical interest are usually the perturbations of the Gaussian solutions of Eq. (11), which are available in closed form.¹⁸

$$A_0(\mathbf{r}; z) = \frac{(w_f^2 + 2i[z_0 - z_f])}{(w_f^2 + 2i[z - z_f])} \exp \left[- \frac{r^2}{(w_f^2 + 2i[z - z_f])} \right], \quad (12)$$

where w_f is the beam radius at the focal point z_f with respect to the transformed units and the solution is normalized so that $A_0(0; z_0) = 1$. In experimental settings usually a collimated beam is focused at a desired location with a convex lens. It is convenient to let z_0 be the lens location. The solution given by Eq. (12) is determined by its value at one point on the beam.

While Eq. (10) admits the two-dimensional version of the representation given by Eq. (2) and thus, can be solved in principle by using Eq. (4) by appropriate adjustments, in numerical applications an evaluation of the propagator must be carried out by restricting the integration to a finite interval. In cases of rapidly diverging or converging beams, the interval where the values of the amplitude are significant varies accordingly. This necessitates a variable mesh structure. A mesh structure conforming to the characteristic curves of Eq. (11) corresponding to the solution given by Eq. (12) can be used to circumvent the difficulty.¹⁴ This amounts to having a fixed mesh structure with respect to the variable $\rho = \mathbf{r} / w(z)$, where

$$w(z) = w_f \left[1 + 4 \frac{(z - z_f)^2}{w_f^2} \right]^{1/2}$$

is the beam radius at z . With respect to the variables

$$\rho = \mathbf{r} / w(z) \text{ and } \xi = \frac{1}{2} \tan^{-1} \left[\frac{2(z - z_f)}{w_f^2} \right],$$

Eq. ((10) assumes the following form:

$$i \frac{\partial A}{\partial \xi} = - \frac{1}{2} \nabla_{\rho}^2 A + 2i \tan(2\xi) (\rho \bullet \nabla_{\rho}) A + w(\xi)^2 w_0 \hat{V}(I) A, \quad (13)$$

where $w(\xi) = w(z(\xi)) = w_f / \cos(2\xi)$. With the representation

$$A = \cos(2\xi) \exp \left[i \rho^2 \tan(2\xi) \right] B, \quad (14)$$

B satisfies the equation

$$i \frac{\partial B}{\partial \xi} = - \frac{1}{2} \nabla_{\rho}^2 B + V B, \quad (15)$$

where

$$V = \left(2 \rho^2 + w^2(\xi) w_0 \hat{V} \right).$$

The solution of the unperturbed equation corresponding to A_0 is given by

$$B_0(\rho, \xi) = \frac{\exp [2i (\xi_0 - \xi)]}{\cos (2\xi_0)} \exp [- \rho^2]. \quad (16)$$

Eq. (15) is the same as Eq. (1) with slight adjustment to the potential term. This transformation essentially eliminates any convergence or divergence of the beam. The solution can be obtained by first evaluating B by proper substitutions in Eq. (4) and then A , which is more convenient whether there are rapid variations in the beam radius or not and better suited to all numerical schemes.

3. Numerical scheme

It is clear from the analysis of Sec. 2 that in order to treat all of the above cases it is sufficient to develop a numerical procedure to evaluate one-dimensional $P(\varepsilon)$ given by Eq. (3) expressed as

$$(P(\varepsilon)u(\tau))(x, \tau + \varepsilon) = (M(\varepsilon)[P'(\varepsilon) M(\varepsilon)u(\tau)])(x, \tau + \varepsilon), \quad (17)$$

where $M(\varepsilon)$ is the operation of multiplication defined by

$$(M(\varepsilon)h)(x) = \exp[ix^2/(2\varepsilon)] h(x)$$

and $P'(\varepsilon)$ is a Fourier transform:

$$(P'(\varepsilon)g(\tau))(x, \tau + \varepsilon) = \frac{1}{\sqrt{2\pi i \varepsilon}} \int_{-\infty}^{\infty} dx' \exp\left[-\frac{i}{\varepsilon} x x'\right] g(x', \tau). \quad (18)$$

Among other procedures available to calculate the Fourier transforms, they can be evaluated also by the methods of *FFT*. The resulting scheme is quite convenient if the sets of grid points for the functions g and f are reciprocals of each other. To determine the evolution of the solutions for the case presently under consideration, it is desirable to have the same input and output grid structures with respect to the space variable. This is possible but cumbersome and compromises accuracy, which is therefore not preferred normally. Major complicating feature is the rapidly oscillating integrand resulting from a small value of ε , which is desirable for accuracy. It is shown below that this disadvantage can be exploited by constructing a suitable grid structure to reduce the operation of $P'(\varepsilon)$ essentially to an *FFT*⁻, where the superscript refers to the sign of the exponent in the Fourier transform.

The integral in Eq. (18) can be approximated by restricting the interval of integration to $[-a, a]$ with a sufficiently large but finite value of a . The value of the integrand outside this interval is assumed to be negligible. By the mid point integration with equispaced grid points, $\{a(2n/N - 1)\}_{n=0}^N$, the integral is further approximated by

$$\begin{aligned} (P'(\varepsilon)g(\tau))\left(a(2\frac{n}{N}-1), \tau + \varepsilon\right) &= \frac{2a}{N\sqrt{2\pi i\varepsilon}} \sum_{m=1}^N g\left(a(2\frac{n}{N}-1), \tau\right) \times \\ &\quad \exp\left[-\frac{4a^2}{\varepsilon}i\left(\frac{nm}{N^2}-\frac{n+m}{2N}+\frac{1}{4}\right)\right], \quad (19) \\ n &= 1, 2, \dots, (N-1); \quad (P'(\varepsilon)g(\tau))(a, \tau + \varepsilon) = (P'(\varepsilon)g(\tau))(-a, \tau + \varepsilon) = 0. \end{aligned}$$

With $a = \sqrt{\pi N \varepsilon / 2}$, Eq. (19) can be solved efficiently by *FFT*. Since N is typically a large integer and ε , a small number, this condition yields a moderate value of a , which is commensurate with the decay properties of a large class of solutions. Substitution from Eq. (19) in Eq. (17) yields

$$\begin{aligned} (P(\varepsilon)g(\tau))\left(a(2\frac{n}{N}-1), \tau + \varepsilon\right) &= \frac{2a}{N\sqrt{2\pi i\varepsilon}} \exp\left[\frac{\pi i}{N}(n-1)^2\right] \times \\ &\quad \sum_{m=1}^N g\left(a(2\frac{m}{N}-1), \tau\right) \exp\left[-2\pi i \frac{(n-1)(m-1)}{N}\right] \exp\left[\frac{\pi i}{N}(m-1)^2\right], \quad (20) \\ n &= 1, 2, \dots, (N-1); \quad u[-a, \tau + \varepsilon] = u[a, \tau + \varepsilon] = 0. \end{aligned}$$

Eq. (20) is pre-eminently suitable for the standard *FFT* routines.

While the condition $a = \sqrt{\pi N \varepsilon / 2}$ can be implemented to accommodate various practical situations with other choices, the following values of the parameters were used for the present calculations for convenience:

$$a = \sqrt{2^\gamma \pi}, \quad \gamma = 0, 1, 2, \dots, \quad \varepsilon = \frac{1}{2^\nu}, \quad \nu = 1, 2, \dots,$$

which determine N :

$$N = 2^\mu, \quad \mu = (\nu + \gamma + 1).$$

The interval of integration can be selected first to be sufficiently large by setting a appropriately. Then ε can be set sufficiently small to ensure the accuracy of the propagation with respect to τ . The grid structure for the variable x is determined by a and ε . Rescaling of the variables τ and x can also be used to increase the flexibility of the method, particularly to accommodate pre-set intervals for the variables.

By the present method, the solution accurate up to $o(\varepsilon^2)$ is given by

$$u(\tau + 2\varepsilon) = M P'(\varepsilon) M \exp[2i\varepsilon V(\tau)] M P'(\varepsilon) M u(\tau). \quad (21)$$

In comparison, the widely used split-step scheme based on FFT defines the solution up to the same degree of accuracy by

$$u(x, \tau + 2\varepsilon) = FFT^+ \left(\exp[-i\varepsilon k^2] FFT^- \left\{ \exp[2i\varepsilon V] FFT^+ \left(\exp[-i\varepsilon k^2] FFT^- u(\tau) \right) \right\} \right). \quad (22)$$

The operations of FFT^+ and FFT^- in Eq. (22) can be interchanged without affecting the result. Thus, except for elementary operations of multiplication, use of Eq. (21) reduces the number of numerical operations required to about half in comparison with Eq. (22).

4. Examples

4.1. Spatial solitons.

The nonlinear Schrödinger equation

$$i \frac{\partial u}{\partial z} + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + (\alpha |u|^\sigma + \eta |u|^{2\sigma}) u = 0, \quad (23)$$

with $\alpha\gamma < 0$ is known to possess stable one soliton solutions given by

$$u(x, z) = \left[\frac{A}{\cosh(Dx) + B} \right]^{1/\sigma} e^{i\beta z}.$$

$$A = (2 + \sigma)B\beta/\alpha, \quad D = \sigma\sqrt{2\beta},$$

where

$$B = \pm \left[1 + \frac{(2 + \sigma)^2 \eta}{(1 + \sigma) \alpha^2} \beta \right]^{-1/2},$$

$\beta > 0$ is otherwise arbitrary and the multiplier of the phase factor in the solution u is assumed to be positive.¹³ Although among the simplest cases of Eq. (1), Eq. (23) is sufficiently interesting to warrant consideration.

Convergence of the numerical solutions obtained by the present method for this case is illustrated in Figure 1. The plotted values correspond to $\alpha=1$, $\beta=1/2$, $\eta=-1/4$ and $\sigma=2$, for $z=25$ starting with the initial condition at $z=0$. With step size $(2\varepsilon)=0.5$ and the corresponding 128 grid points with respect to the x variable over a spread of $4\sqrt{\pi}$,

calculations produce the solution almost exactly. This example is representative of several other similar calculations with different sets of defining parameters.

Figure 1. Convergence of the present procedure for (1+1)-soliton solution. Half interval $a = 4\sqrt{\pi}$. Plot compares the numerical values with $\Delta z = 2\varepsilon = 1.0$, $N = 64$ (dashed line) and $\Delta z = 2\varepsilon = 0.5$, $N = 128$ (dotted line) with the exact solution (solid line).

4.2. Wave in lossy-gain medium.

A wave propagating in a realistic PBG material suffers absorption losses and energy dissipation due to vibrations. Also, doping the material may result in loss or gain. For these reasons, there is interest in the properties of the waves in lossy-gain mediums. The governing equation for this case in one dimension in the effective mass approximation near the upper band edge reads¹¹

$$i \frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + (i\Gamma_L - \alpha_1) u + (n_{NL} + i\Gamma_{NL}) |u|^2 u = 0, \quad (24)$$

with appropriate scaling of the x variable. Eq. (24) has solitary wave solutions given by

$$u(x,t) = A \operatorname{sech}[\alpha(t - x/v)] \exp[i(vx + \phi(x,t))],$$

where

$$\phi(x,t) = b \log[\operatorname{sech}(\alpha(t - x/v))], \quad A = \sqrt{-3\Gamma_L/2\Gamma_{NL}}, \quad \alpha = \sqrt{-\Gamma_L/b}$$

and

$$b = (\alpha_1/\Gamma_L) \pm \sqrt{1 + ([\alpha_1 + v^2/2]/\Gamma_L)^2}.$$

For a physically admissible solution, A and α are real and the velocity v is an arbitrary parameter.

Figure 2. Intensity profiles of a solitary wave moving with unit velocity in a lossy-gain medium. Plot shows the initial profile (solid line), exact profile at time = 2.5 (dashed line) and the calculated values with half interval $a = 4\sqrt{\pi}$, $\Delta t = 2\varepsilon = 0.25$, $N = 256$ (dotted line).

The results for this case are illustrated in Figure 2 with $v=1$ at time $t=2.5$. Various physically reasonable combinations of the parameters produced qualitatively similar results. Further refinement of the mesh produced results almost identical to the exact values.

4.3. Waves in PBG material.

A class of problems representing propagation in the photonic bandgap material in slowly varying envelope approximation in an appropriate Lorentz frame is described by the following coupled set of equations:

$$\begin{aligned} i \frac{\partial E_1}{\partial \tau} + \frac{1}{2} \frac{\partial^2 E_1}{\partial x^2} + \delta E_1 + \beta E_2 + \frac{2\alpha}{3} [|E_1|^2 + 2 |E_2|^2] E_1 &= 0 \\ i \frac{\partial E_2}{\partial \tau} - \frac{1}{2} \frac{\partial^2 E_2}{\partial x^2} - \delta E_2 - \beta E_1 - \frac{2\alpha}{3} [|E_2|^2 + 2 |E_1|^2] E_2 &= 0 \end{aligned} \quad (25)$$

(See, e.g., Eqs. 2.24a and 2.24b of Ref. [12]). Eq. (25) is expressed as Eq. (7) with

$$v = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}, \quad V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix},$$

where

$$\begin{aligned} V_{11} &= \left(\delta + \frac{2\alpha}{3} [|E_1|^2 + 2 |E_2|^2] \right), & V_{12} &= \beta, \\ V_{12} &= -\beta, & V_{21} &= -\left(\delta + \frac{2\alpha}{3} [|E_1|^2 + 2 |E_2|^2] \right), \end{aligned}$$

$\zeta_1 = 1$ and $\zeta_2 = -1$. Approximate solution in the present framework is given by Eq. (8). An exact solution for this case is not available.

Eq. (25) is known to possess soliton solutions, which can be expressed as combinations of soliton basis functions, the most elementary one being $[Sech(x)]$. Figure 3 shows the

profile of a beam with initial value equal to $[Sech(x)]$ for the parameter values as indicated. The mesh structure corresponds to $a = 4\sqrt{\pi}$, $\Delta t = 2\varepsilon = 0.0625$, $N = 512$. However, little change was observed from $N = 256$ onwards. Several other profile calculations showed similar behavior.

Figure 3. Soliton-like signal in PBG crystal. Plot shows the initial intensity (solid line), intensity with $\beta = \delta = 0.1$, $\alpha = 0.2$, at time = 1.0 (dashed line) and intensity with $\beta = \delta = 0.1$, $\alpha = 0.5$, at time = 1.0 (dotted line). Parameter values $\beta = \delta = 0.1$, $\alpha = 0.35$ produced intensity close to the initial.

4.4. Converging beam

Propagation of an optical beam in a dielectric medium is described by Eq. (9) with the solution A coinciding with A_0 as given by Eq. (12) at the lens location z_0 . With z_f being

the location of the focal point, w_f and $w(z_0)$ determine each other. The potential term in case of laser fabrication of materials is given by¹⁻⁶

$$\hat{V} = - \left[n_2 k I - \frac{2\pi e^2 N_e(I)}{k m_e c^2} \right]. \quad (26)$$

The first term describes the self-focusing effect in terms of the Kerr coefficient n_2 , which by itself is of interest in some cases.⁷ The second term describes the generation and effect of plasma resulting from the interaction of high intensity radiation with the material in terms of the electron charge e , electron mass m_e and the speed of light c in the gas. Free electron density N_e induced by the laser pulse by field ionization is determined by the rate equation:

$$\frac{\partial N_e(I(t))}{\partial t} = N_0 R_{ion}(I(t)), \quad (27)$$

where N_0 is the number density of the neutral air molecules, R_{ion} is the ionization rate and t is the retarded time. In Sec. 2, this case was reduced to Eq. (15).

Figure 4 shows the self-focusing effect and the impact of the plasma generated on an initially Gaussian beam propagating through air. The collimated beam at the lens coincides with Eq. (12) with a multiplicative factor of $\exp \left[-\kappa(t/\tau_p)^2 \right]$ with respect to time t , where $\kappa = 4\log(2)$ and τ_p is the pulse width. The optical signal was assumed to be focused with a lens of focal length of 21 mm with beam radius at the focal point equal to 1 μm with corresponding radius at the lens equal to 5.18 mm. The wavelength of the radiation was

taken to be 775 nm , pulse width of the laser equal to 150 fs and the pulse energy equal to $200 \text{ }\mu\text{J}$ with corresponding peak input power at the lens equal to $2.97 \times 10^{13} \text{ W/m}^2$. These parameters correspond to a Clark-MXR CPA 2010 femtosecond laser with Ti:Sapphire regenerative amplifier. The Kerr coefficient of air is $5.0 \times 10^{-23} \text{ m}^2/\text{W}$. The rate R_{ion} was calculated from a polynomial fit to its experimental values¹⁹ for O_2 and N_2 assuming the atmosphere to contain 80% O_2 and 20% N_2 . The plots shown were calculated with $\gamma = 1$, $\nu = 5$ and the corresponding 128×128 grid. Almost no change was observed by refining the grid further.

The values plotted in Figure 4 show the fluence at the focal point normalized in units of the peak input fluence, in vacuum, with only the self-focusing effect included and with both, the self-focusing and the effect of plasma included, against the normalized radial coordinate ρ . The numerical results for propagation in vacuum agree with the corresponding exact profile given by Eq. (12). Self focusing effect increases with increase in intensity. The effect of plasma is proportionate to the self-focusing effect but opposite in sign. Consequently, the resulting profile shows a dip at the central position where the original intensity enhanced by self-focusing is very high, thereby yielding high-density plasma. In addition to the dip at the center, beam is distorted elsewhere as well as a consequence. These results are commensurate with other similar calculations and observations.¹⁻⁶ Neon with the Kerr coefficient equal to $1.0 \times 10^{-24} \text{ m}^2/\text{W}$ showed little self-focusing effect and the effect of Helium with Kerr coefficient equal to $3.5 \times 10^{-25} \text{ m}^2/\text{W}$ was found to be unnoticeable at this pulse energy and the other parameter values, with commensurate impact of plasma.

Figure 4. Self-focusing and plasma effects of air on a Gaussian signal. Plot shows fluences normalized in units of peak input fluence in vacuum (dashed line) with self-focusing term (dotted line) and with resulting plasma effects included (solid line).

5. Concluding remarks

A numerical scheme is developed in the present article to evaluate the propagator obtained by the path integral method in quantum mechanics to represent the time evolution of the solutions of the time-dependent Schrödinger type equations. Also, the propagation of an optical beam is formulated to eliminate the difficulties caused by rapid variations in its radius. The resulting equation is more convenient to solve by other numerical schemes as well as by the present. The numerical procedure developed in the present article can be implemented by the standard *FFT* routines conveniently and more efficiently than the other methods used for this purpose to obtain the solutions accurately. The treatment covers a

variety of physical phenomena described by the linear and nonlinear, single and coupled, Schrödinger type equations. This includes the soliton-like and other solutions in PBG-material, material processing with laser beams and quantum mechanical phenomena. The analysis can be extended to cover other differential equations.

The numerical procedure is tested on several problems, some of which are exactly solvable. Some of the applications considered here have been treated in literature by quite cumbersome analytical and numerical schemes. In all cases, the convergence of the present scheme is found to be quite rapid and thus, the solutions are obtained very efficiently.

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