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Shape Sensitivity Analysis of Unsteady Laminar Flow Past a Cylinder in Ground Proximity

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This paper presents the application of a general shape sensitivity equation method to unsteady laminar flows. The formulation accounts for complex parameter dependence and is suitable for a wide range of problems. The methodology is applied to the flow past a cylinder in ground proximity. The solution of the continuous sensitivity equation method (SEM) is compared with the gradient computed as a finite difference between two flow solutions. This study investigates the ability of the sensitivity equation method to anticipate the unsteady flow response: vortex shedding damping when the ground to cylinder gap decreases and the amplification of shedding with increasing the gap size.

I. Introduction

Sensitivity analysis is a relatively new and powerful tool in computational fluid dynamics. A sensitivity (the derivative of the solution with respect to a parameter) indicates how a dependent variable reacts to variations of a design parameter. Sensitivity information finds many uses ranging from driving optimization algorithms, to fast evaluation of nearby flows or to produce uncertainty estimates of the solution. Sensitivities also find applications in flow control due to their ability to indicate the flow response to design parameter changes. In all cases cost-effectiveness is achieved because sensitivities are obtained at a fraction of the cost of computing the flow.

There are several means of computing flow sensitivities: finite differences of flow solutions, the complex step method,¹ automatic differentiation,² and sensitivity equation methods.³⁻⁵ The first option is costly because the problem must be solved for two or more values of each parameter of interest. Furthermore, technical problems arise because non matching meshes are obtained for different values of a shape parameter. The complex-step method is code invasive: it requires a complete rewrite of the software in complex variables. While this can be automated, it has a significant impact on performance. Automatic differentiation is equivalent to differentiating the discrete equations to generate a system of equations for the discrete sensitivities. It is powerful because it automatically generates the code for calculating sensitivities. In many cases, implementation requires human intervention to ensure efficiency of the code. Approaches to calculating sensitivities also differ depending on the order of the operations of approximation and differentiation. In the *discrete* sensitivity equation approach, the total derivative of the flow approximation with respect to the parameter is calculated,⁶ whereas in the *continuous* sensitivity equation method (SEM) one differentiates the continuum equations to yield differential equations for the continuous sensitivities.³ See Hien et al.⁷ for a discussion of the two approaches. We have adopted the latter approach.

Sensitivity analysis is a more advanced field in solid mechanics than in fluid dynamics. Indeed, textbooks have been written on sensitivity analysis of structures.^{6,7} To our knowledge there is only one book on sensitivity analysis of flow problems.⁴ It is recent and more specialized than structural mechanics books. Gunzburger⁸ discusses sensitivity analysis in the context of flow control and optimization.

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Automatic differentiation for first-order flow sensitivities is discussed by Sherman *et al.*⁹ and Putko *et al.*² Continuous SEMs may be found in Godfrey and Cliff,^{10,11} Borggaard and Burns,³ Limache¹² and Turgeon *et al.*¹³ for aerodynamics applications. Application to heat conduction is reported by Blackwell *et al.*¹⁴ Sensitivities for incompressible flows with heat transfer may be found in several references.^{5,15,16} Sensitivity analysis for turbulence models is detailed in the works by Godfrey and Cliff¹¹ and Turgeon *et al.*¹⁷ Solution of the sensitivity equations for the transient incompressible flow of non-Newtonian fluids is presented by Ilinca and Hétu.¹⁸ A wide variety of flow regimes were treated by the authors.^{5,15-17} This body of work has shown that sensitivities provide an enriched basis of information on which to develop an improved understanding of complex flow problems. The work presented here is an extension to unsteady flows of the shape sensitivity methodology presented by Duvigneau and Pelletier.¹⁹ It is also an extension to shape parameters of the unsteady SEM by Hristova *et al.*,²⁰ and Ilinca *et al.*²¹ for laminar flows.

The paper is organized as follows. First, we present the equations describing time-dependent laminar flow along with their boundary and initial conditions. The shape sensitivity equations and their boundary/initial conditions are then described in detail. The approach is applied to the flow around square and circular cylinders in ground proximity. The paper will investigate the ability of sensitivities to predict important changes in the flow response such as the damping of the vortex shedding observed with a decrease of the ground to cylinder gap and the increased amplitude of the unsteadiness when the gap increases. The paper ends with conclusions.

II. Flow equations

The flow regime of interest is modeled by the momentum and continuity equations:

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \mathbf{f} + \nabla \cdot \left[\mu \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T \right) \right] \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0 \quad (2)$$

where ρ is the density, $\mathbf{u}$ is the velocity, p is the pressure, μ is the viscosity, t represents time and $\mathbf{f}$ is a body force. The above system is closed with a proper set of initial conditions

$$\mathbf{u}(\mathbf{x}, t = 0) = \mathbf{U}_0(\mathbf{x}) \text{ in } \Omega \quad (3)$$

and Dirichlet and Neumann boundary conditions

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{U}_D(\mathbf{x}, t) \text{ on } \Gamma_D. \quad (4)$$

$$\mathbf{t} = [-p\mathbb{I} + 2\mu\boldsymbol{\gamma}(\mathbf{u})] \cdot \hat{\mathbf{n}} = \mathbf{F}^N \text{ on } \Gamma_N \quad (5)$$

where $\mathbf{U}_D$ is the value of the velocity imposed on the boundary Γ_D , $\mathbb{I}$ is the identity tensor, $\boldsymbol{\gamma}(\mathbf{u}) = (\nabla \mathbf{u} + \nabla \mathbf{u}^T)/2$ is the shear rate tensor and $\mathbf{F}^N$ is the imposed boundary value of the surface traction force $\mathbf{t}$.

III. Sensitivity Equations

The continuous sensitivity equations (CSE) are derived formally by implicit differentiation of the flow equations (1) and (2) with respect to parameter a . We treat the variable $\mathbf{u}$ as a function of space, time and of the parameter a . This dependence is denoted as $\mathbf{u}(\mathbf{x}, t; a)$. Defining the flow sensitivities as the partial derivatives $\mathbf{s}_u = \frac{\partial \mathbf{u}}{\partial a}$ and $s_p = \frac{\partial p}{\partial a}$, and denoting the derivatives of the fluid properties and other flow parameters by a ($'$), differentiation of equations (1) and (2) yields

$$\rho' \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) + \rho \left(\frac{\partial \mathbf{s}_u}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{s}_u + \mathbf{s}_u \cdot \nabla \mathbf{u} \right) = -\nabla s_p + \mathbf{f}' + \nabla \cdot \left[\mu' \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T \right) + \mu \left(\nabla \mathbf{s}_u + (\nabla \mathbf{s}_u)^T \right) \right] \quad (6)$$

$$\nabla \cdot \mathbf{s}_u = 0. \quad (7)$$

A. Initial and Boundary Conditions

Initial conditions for the sensitivity equations are obtained by implicit differentiation of equation (3)

$$\mathbf{s}_u(\mathbf{x}, t = 0) = \frac{\partial U_0}{\partial a}(\mathbf{x}) \quad \text{in } \Omega, \quad (8)$$

Dirichlet and Neumann boundary condition are obtained in a similar manner. However, if a is a shape parameter, the position of the boundary is parameter dependent. Therefore, the differentiation must account for the dependence on a of both boundary value and boundary location. The boundary conditions for the CSE are written as:

$$\mathbf{s}_u = \frac{\partial U_D}{\partial a} - \nabla \mathbf{u} \cdot \frac{\partial \mathbf{x}}{\partial a} \quad \text{on } \Gamma_D, \quad (9)$$

$$[-s_p \mathbb{I} + 2(\mu \gamma(\mathbf{s}_u) + \mu' \gamma(\mathbf{u}))] \cdot \hat{\mathbf{n}} = \frac{\partial F^N}{\partial a} - \left\{ \nabla \cdot [-p \mathbb{I} + 2\mu \gamma(\mathbf{u})] \cdot \frac{\partial \mathbf{x}}{\partial a} \right\} \cdot \hat{\mathbf{n}} - [-p \mathbb{I} + 2\mu \gamma(\mathbf{u})] \cdot \frac{\partial \hat{\mathbf{n}}}{\partial a} \quad \text{on } \Gamma_N \quad (10)$$

As can be seen, for a shape parameter, first derivatives of the flow appear in Dirichlet boundary conditions and second derivatives are required for Neumann boundary conditions. This introduces numerical difficulties when solving CSE, since approximate boundary conditions are used. Accuracy of sensitivity boundary conditions is improved by extracting the normal derivatives from the auxiliary fluxes computed on the boundary (see the next section).

B. Evaluation of velocity gradient at Dirichlet boundary nodes

For simplicity we restrict ourselves to the case where Dirichlet conditions are specified on the parameter dependent boundary segment. For this case flow gradients at the wall are needed to evaluate boundary conditions via equation (9). To improve accuracy we express the derivatives in the normal and tangential directions as follows

$$\frac{\partial \mathbf{u}}{\partial n} = \frac{\partial \mathbf{u}}{\partial x} n_x + \frac{\partial \mathbf{u}}{\partial y} n_y + \frac{\partial \mathbf{u}}{\partial z} n_z \quad (11)$$

$$\frac{\partial \mathbf{u}}{\partial t_1} = \frac{\partial \mathbf{u}}{\partial x} t_{1x} + \frac{\partial \mathbf{u}}{\partial y} t_{1y} + \frac{\partial \mathbf{u}}{\partial z} t_{1z} \quad (12)$$

$$\frac{\partial \mathbf{u}}{\partial t_2} = \frac{\partial \mathbf{u}}{\partial x} t_{2x} + \frac{\partial \mathbf{u}}{\partial y} t_{2y} + \frac{\partial \mathbf{u}}{\partial z} t_{2z} \quad (13)$$

where $\hat{\mathbf{n}}$ is a unit vector normal to the boundary and $\hat{\mathbf{t}}_1 = (t_{1x}, t_{1y}, t_{1z})$, $\hat{\mathbf{t}}_2 = (t_{2x}, t_{2y}, t_{2z})$ are two orthogonal unit vectors tangent to the boundary. On boundaries with homogeneous Dirichlet condition the tangential derivatives are zero, $\frac{\partial \mathbf{u}}{\partial t_1} = 0$, $\frac{\partial \mathbf{u}}{\partial t_2} = 0$, and we need only determine the velocity derivative in normal direction. We do this by using the *auxiliary* traction force $\mathbf{t}_b$ ²² on the boundary

$$\int_{\Omega} \left(\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} \right) \mathbf{w} d\Omega - \int_{\Omega} p \nabla \cdot \mathbf{w} + \int_{\Omega} \mu \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T \right) : \nabla \mathbf{w} d\Omega - \int_{\Omega} f \mathbf{w} d\Omega_P = \int_{\Gamma} \mathbf{t}_b \mathbf{w} d\Gamma. \quad (14)$$

where

$$\mathbf{t}_b = [-p \mathbb{I} + 2\mu \gamma(\mathbf{u})] \cdot \hat{\mathbf{n}} \quad (15)$$

For incompressible flows and homogeneous Dirichlet conditions equation (15) reduces to

$$\mathbf{t}_b = -p \hat{\mathbf{n}} + \mu \frac{\partial \mathbf{u}}{\partial n} \quad (16)$$

which we use to extract the normal derivative of the velocity at the boundary as

$$\frac{\partial \mathbf{u}}{\partial n} = \frac{1}{\mu} (\mathbf{t}_b + p \hat{\mathbf{n}}) \quad (17)$$

Finally, cartesian derivatives of the velocity are obtained by solving equations (11), (12) and (13) given the normal and tangential derivatives. Improved accuracy is achieved by projecting the velocity gradient into a divergence free space. This is done by using the incompressibility condition as a constraint when solving equations (11), (12) and (13).

C. Normal Velocity at Dirichlet Boundary Points

For incompressible flows, the boundary conditions for velocity sensitivities satisfy strict relationships on Dirichlet boundaries. Without loss of generality we restrict ourselves to the case of homogeneous Dirichlet conditions. In this case the shape sensitivity of the velocity is always tangent to the boundary. This property is very useful for assessing the accuracy of the computed boundary conditions for the sensitivities. To prove that, let's start with equation (9) and $U_D = 0$:

$$\mathbf{s}_u = -\nabla \mathbf{u} \cdot \frac{\partial \mathbf{x}}{\partial a} \quad (18)$$

which has the scalar components:

$$s_u = -\left[\frac{\partial u}{\partial x} \frac{\partial x}{\partial a} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial a} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial a} \right] \quad (19)$$

$$s_v = -\left[\frac{\partial v}{\partial x} \frac{\partial x}{\partial a} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial a} + \frac{\partial v}{\partial z} \frac{\partial z}{\partial a} \right] \quad (20)$$

$$s_w = -\left[\frac{\partial w}{\partial x} \frac{\partial x}{\partial a} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial a} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial a} \right] \quad (21)$$

The component of the velocity sensitivity normal to the boundary is given by:

$$-\mathbf{s}_u \cdot \hat{\mathbf{n}} = \left[\frac{\partial u}{\partial x} n_x + \frac{\partial v}{\partial x} n_y + \frac{\partial w}{\partial x} n_z \right] \frac{\partial x}{\partial a} + \left[\frac{\partial u}{\partial y} n_x + \frac{\partial v}{\partial y} n_y + \frac{\partial w}{\partial y} n_z \right] \frac{\partial y}{\partial a} + \left[\frac{\partial u}{\partial z} n_x + \frac{\partial v}{\partial z} n_y + \frac{\partial w}{\partial z} n_z \right] \frac{\partial z}{\partial a} \quad (22)$$

where $\hat{\mathbf{n}} = (n_x, n_y, n_z)$ is the direction normal to the boundary and the velocity gradients satisfy the incompressible condition (2):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (23)$$

Consider now the first term in brackets on the right hand-side of (22) and replace $\frac{\partial u}{\partial x}$ by $-(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z})$, to obtain:

$$\frac{\partial u}{\partial x} n_x + \frac{\partial v}{\partial x} n_y + \frac{\partial w}{\partial x} n_z = \left(\frac{\partial v}{\partial x} n_y - \frac{\partial v}{\partial y} n_x \right) + \left(\frac{\partial w}{\partial x} n_z - \frac{\partial w}{\partial z} n_x \right) \quad (24)$$

Because $\mathbf{u}$ vanishes on the boundary, the derivative of the velocity along any tangent direction: $\nabla \mathbf{u} \cdot \hat{\mathbf{t}} = 0$. Thus, the two terms in the right hand-side of equation (24) vanish, because $(n_y, -n_x, 0)$ and $(n_z, 0, -n_x)$ are two tangent directions and therefore the derivatives of v and w along these directions are zero. A similar treatment applies to the other two terms of eq. (22) so that the final result is obtained:

$$\mathbf{s}_u \cdot \hat{\mathbf{n}} = 0 \quad (25)$$

In the case of a non-homogeneous Dirichlet boundary condition the normal component of the velocity sensitivity is non-zero but can be determined in terms of the velocity boundary condition. In any case, the normal component is known and can be used as an error estimator to test the accuracy of the computed sensitivity boundary conditions.

IV. Implementation

The flow equations and the CSE are solved on three-dimensional meshes by a Streamline-Upwind Petrov Galerkin (SUPG) finite element method.²³ Time is discretized by an implicit Euler scheme. The equations are linearized with Newton's method and discretized with a 4-node tetrahedral element with linear interpolants for both velocity and pressure. The same element is used to solve the sensitivity equations. Element matrices are constructed using a numerical Jacobian technique and assembled in a compressed sparse row format. Global systems for the flow and each of its sensitivities are solved by stabilized BiCG iterative methods.

V. Numerical Results

A. Flow around a square cylinder in ground proximity

Figure 1. Flow around a square cylinder in ground proximity: Definition and mesh.

This problem was the object of several experimental and numerical studies especially for turbulent flow.²⁴⁻²⁶ The computational domain and boundary conditions for this problem are shown on Figure 1(a). Because the problem is two-dimensional only a slab was meshed with 3D tetrahedral elements. The mesh is shown on Figure 1(b) and was designed to provide adequate resolution of the flow and its sensitivities. The inflow velocity U_0 is considered uniform. The initial conditions are obtained from a steady state solution of the flow and sensitivity equations. The Reynolds number $Re = \rho U_0 d / \mu$ is set to 100.

Figure 2. Flow around a square at $s = 0.5D$ from the wall: Steady-state flow.

Figure 3. Flow around a square at $s = 2D$ from the wall: Symmetrical vortices at $t = 5s$.

The flow past the square cylinder induces steady-state recirculating vortices for small distances to the wall. When the distance to the wall increases above a critical value vortex shedding is triggered behind the cylinder resulting in the well known Karman vortex street. Figure 2 illustrates vorticity contours of the solution obtained for $s = 0.5D$. The solution shown was obtained after $t = 120s$ of simulation and no unsteady effects are apparent. The vorticity field for $s = 2D$ at an early time is shown in Figure 3. The vortices behind the cylinder are almost symmetrical and the unsteady oscillations are very small. At latter times the flow becomes unsteady and the Karman vortex street develops behind the cylinder. Figure 4 shows vorticity contours over a time interval close to the vortex shedding period. Vortices form at the rear corners of the square cylinder and are transported downstream in the wake. The vortices formed at the lower corner are slightly damped by the presence of the wall.

This flow presents special challenges to sensitivity analysis due to singularities induced by the sharp corners. To avoid this difficulty we chose to replace the sharp corners by rounded edges with radius equal to one tenth the size of the square obstacle. This problem will lead to better results and smoother sensitivity fields than the flow around the square cylinder with sharp corners since it avoids the effect of the singularities present in the flow solution at the corners.

Figure 4. Flow around a square at $s = 2D$ from the wall: Karman vortex street at later times.

The sensitivity of the velocity (s_u, s_v) at $x = D$, $x = 2D$, and $x = 4D$ is shown in Figure 5 for a distance to the wall $s = D$. The CSE solution is compared with a finite difference approximation of two flows with slightly different wall gap (FD in Figure 5). For this, the flow is solved for two values of the gap size $s = D - \delta s$ and $s = D + \delta s$ where $\delta s \ll D$ (here we take $\delta s = 0.001D$). The finite difference approximation of the sensitivity is computed as

$$(s_u)_{FD} = \frac{u(D + \delta s, \mathbf{x}, t) - u(D - \delta s, \mathbf{x}, t)}{2\delta s} \quad (26)$$

The accuracy of the FD sensitivity is of order $\mathcal{O}(\delta s^2)$. To minimize the effects of mesh changes (mesh sensitivity) on the solution the domain is discretized by a multi-block mesh such that small changes of mesh points are confined to a region near the cylinder. Finite differences of the flow are computed only where the mesh does not change with the parameters. As can be seen from Figure 5 the CSE provides sensitivities that are in excellent agreement with the finite difference approximation indicating that the sensitivity equation method performs well. It also indicates that the boundary flow gradients used in the sensitivity boundary conditions are computed accurately.

Computed sensitivities are compared to finite differences in Figure 6 for the unsteady problem at $t = 80$. Here again the agreement is very good. Small differences between the two solutions are observed for u at $x = 4D$ and for v at $x = 2D$ and $x = 4D$. This indicates that the numerical errors are amplified in time. However, the SEM is able to accurately predict the sensitivity of the flow with respect to the distance to the wall.

B. Flow around a circular cylinder in ground proximity

The study was continued by computing the flow around a circular cylinder in ground proximity at $Re = 100$. We first verify the accuracy of the sensitivity boundary conditions by evaluating the normal component of the velocity sensitivity at the wall. As shown in section III.C, the boundary conditions for the flow sensitivity

Figure 5. Flow around a square cylinder: Sensitivities with respect to s for steady state flow.

Figure 6. Flow around a square cylinder: Sensitivities with respect to s for the unsteady flow at $t = 80$.

must be tangent to the cylinder surface. In Figure 7 we compare the normal and tangential components of the sensitivity boundary condition. The variable on the horizontal axis represents the angle θ measured from the rear stagnation point ($\theta = 0$ for the point $x = D/2, y = 0$, $\theta = \pi/2$ for $x = 0, y = D/2$, and so on until $\theta = 2\pi$ for $x = D/2, y = 0$). As can be seen, the normal component is very small and it is practically negligible when compared to the tangential component. This indicates that the technique for evaluating boundary conditions for flow sensitivities performs well.

Figure 7. Components of the velocity sensitivity boundary condition on the cylinder surface.

Figure 8. Flow around a circular cylinder at $s = D$ from the wall: Von Karman vortex street.

Transient solutions are first shown for $s = D$. Figure 8 presents vorticity contours for times $t = 104, 106, 108$, and 110 . The results indicate that for this value of s , the vortex street develops quite rapidly. To quantify the effect of s on the formation of the vortex street in the wake of the cylinder, simulations were also carried out for $s = 0.75D$. The vorticity contours are illustrated in Figure 9 for $t = 144$ to $t = 150$, hence at latter times than for the case $s = D$ (Figure 8). For $s = 0.75D$ the vortex street develops more

Figure 9. Flow around a circular cylinder at $s = 0.75D$ from the wall: Initiation of unstable flow.

slowly than for $s = D$ and the amplitude of velocity oscillations is much smaller. This is also shown in Figure 10 which compares the vertical velocity in the two cases at the point $(x = 4D, y = D)$.

Figure 10. Time signal of the vertical velocity at $(x = 4D, y = D)$.

Shape sensitivities with respect to s were computed for the case $s = 0.75D$. The time signals at $(x = 4D, y = D)$ of the flow sensitivities are shown in Figure 11. The solution of the sensitivity equations is compared with the derivative computed by central finite differences with $\delta s = 0.001D$. The flow solution is shown on the left column of the figure and its sensitivity on the right column.

The following observations can be made:

- The period of the sensitivity signals is the same as the period of the flow solution;
- The amplitudes of sensitivities signals are larger and increases faster with time than those of the flow;
- In all cases the computed sensitivities agree very well with the finite difference derivatives.

(a) u

(b) s_u

(c) v

(d) s_v

(e) p

(f) s_p

Figure 11. Flow around a circular cylinder: Sensitivities with respect to s at $(x = 4D, y = D)$.

Figure 12. Amplitude of fluctuations in v and its sensitivity at $(x = 4D, y = D)$.

Figure 12 presents the time variation of the amplitude in the vertical velocity component and that of its sensitivity. Both sets of data are plotted on a logarithmic scale because the scale of the velocity fluctuations is much smaller than that of the sensitivity fluctuations. Again we observe that the amplitude of the sensitivity fluctuations is much larger and increases faster than that of the flow solution. This is an important observation because it indicates that sensitivities appear to react faster than the flow to changes in the parameter values. Sensitivities appear able to foretell the transition from the quasi steady-state solution to the vortex shedding before it becomes visible in the flow signal.

As a final example, the sensitivities are used for fast evaluation of nearby flows. Consider for example what happens to the u -velocity, when a generic parameter a is subject to a variation δa from the reference value a_0 . The Taylor series expansion yields:

$$u(x, y, z, t; s_0 + \delta s) = u(x, y, z, t; s_0) + \left. \frac{\partial u}{\partial s} \right|_{s_0} \delta s + \mathcal{O}(\delta s^2). \quad (27)$$

Using the baseline solution obtained at $s = 0.75D$, we compare the flow estimates from the Taylor series for u , v and p to a full flow analysis at the perturbed values of the parameter. Results for two values of s , one lower ($s = 0.74D$) and the other larger ($s = 0.76D$) than the baseline value are shown in Figure 13. The reconstructed solutions are very close to those obtained by reanalysis at the perturbed value of s . The Taylor series approximations of the flow response are in very good agreement with the CFD reanalysis at early times. Agreement deteriorates very slightly at later times, probably because higher order derivatives in the Taylor series expansion become important with increasing time. These results indicate that the SEM predicts the proper rate of damping toward the steady state solution as s decreases. Taylor series also predict well the increased amplitude of unsteadiness occurring with increased values of s .

VI. Conclusion

A general shape sensitivity equation formulation was developed for the solution of time-dependent incompressible laminar flows. The method is applied to the flow around square and circular cylinders in proximity of a solid wall. The study analyzes the influence of the distance to the wall on the transition from the steady-state flow to vortex shedding behind the cylinder. The sensitivity of the flow is computed and correlated with the flow response when the distance to the wall changes. For the circular cylinder at a distance $s = 0.75D$ from the wall, the amplitude of the sensitivity oscillations increases much faster with time than that of the flow solution. It appears that sensitivities can foretell changes in the flow response: damping of the vortex shedding for decreasing s and amplification of unsteadiness with increasing values of s . This information could be used in a flow control algorithm in order to maintain the flow regime having specific characteristics.

(a) $u(s + \delta s) = u(s) + s_u \delta s$, $\delta s = -0.01D$

(b) $u(s + \delta s) = u(s) + s_u \delta s$, $\delta s = 0.01D$

(c) $v(s + \delta s) = v(s) + s_v \delta s$, $\delta s = -0.01D$

(d) $v(s + \delta s) = v(s) + s_v \delta s$, $\delta s = 0.01D$

(e) $p(s + \delta s) = p(s) + s_p \delta s$, $\delta s = -0.01D$

(f) $p(s + \delta s) = p(s) + s_p \delta s$, $\delta s = 0.01D$

Figure 13. Flow around a circular cylinder: Fast nearby solutions for $s = 0.74D$ (left) and $s = 0.76D$ (right) at $(x = 4D, y = D)$.

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