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An Improved Sampling Approach to Overcome Transient Effect in Staircase Waveform and Its Application

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Abstract—The advantage of the new sampling approach to overcome transient effect in staircase waveform is described. Based on this sampling approach, a general form is developed in this paper. In this form, more samples are used for calculations in the measurement. It provides a better basis to achieve a lower uncertainty as reduced noise impact and more information. A sine-wave experiment shows the advantage of the multi-point approach. When the more frequency components of the measured signal are asked, it faces a matter of unequal-interval sampling. It is found that the aliasing effect of sampling is solvable under the discussed condition. A novel method is developed to perform Fourier transform to extend application of Nyquist Theorem, in which the matrixes to solve the relative equation set are given out in the forms of digital and formula. Experiments for a non-sinusoidal wave validates that the higher frequency components can be analyzed, in which a differential sampling scheme is proposed.

Index Terms—Differential measurement, digital-analog conversion, discrete Fourier transform, non-sinusoidal wave, Nyquist Theorem, sampling, staircase waveform, transient effect.

I. INTRODUCTION

THE application of the staircase waveform produced from a digital-to-analogue conversion in the various fields of the electromagnetic metrology is well accepted and known. However, the transient effect is a main disadvantage, resulting in an intrinsic error which would limit further application. A new sampling approach to overcome the transient effect has been proposed [1, 2] and obtained better results [3]. Its main idea is to discard half of the samples at the transients, and to retain another half samples at the middle of the constant-level step to calculate for the measurements. A differential sampling measurement has been developed for a high-purity sine wave under calibration with the staircase waveform as reference.

This idea can be developed further to a more general form. The more samples at the every step can be used for discrete

Fourier transform (DFT); and the ratio of the discarded to the retaining part may no longer be 1:1 if requirement.

In the fact a section but not a point of waveform at the middle is measured (integral) by a DVM to obtain one reading [1, 2] then DFT with the base functions of the one point at the middle. It holds the equal-interval sampling scheme and obtains an average of the section of waveform. However it neglects the minterm of second order. When the more points are used, it can be so done, every sample is operated of DFT respectively with its own base functions at the relative point, and final to average these Fourier coefficients. It will provide benefit of precise results and more information.

The property of staircase waveform under the proposed form is discussed. An experiment for the sine-wave is undertaken to demonstrate its advantages. A new case appears in application of the Nyquist Theorem due to some samples are discarded. A novel method is hence developed to analyze higher frequency components. It is validated through a non-sine signal experiment.

II. A GENERAL FORM

A. Sampling and DFT Model

Assuming the original sine-wave is

$$y(x) = \sin(x) \quad (1)$$

It is divided with an equal-interval of $h_1 = 2\pi/n$. The samples are $y_i = \sin(ih_1)$ $i = 1, 2, \dots, (n-1)$. From these data an ideal staircase waveform (without impact from quantization and transient) can be written as

$$z(x) = y_i = \sin(ih_1) \quad ih_1 \leq x < (i+1)h_1 \quad (2)$$

Its every step is sampled with another equal-interval of $h_2 = h_1/m = 2\pi/(nm)$. The all samples of the signal (1) and (2) become respectively as

$$y_{i,j} = y[(im+j)h_2] = \sin(imh_2 + jh_2)$$

$$z_{i,j} = z[(im+j)h_2] = \sin(imh_2) = z_i$$

$$j = 0, 1, \dots, (m-1) \quad (3)$$

The same sampling scheme is applied to the base functions produced the relative samples of the $\sin(i, j)$ and the $\cos(i, j)$.

The index of i , ($0 \leq i < n$) is denoted to indicate a step of the staircase waveform, and the index j , ($0 \leq j < m$) to indicate a sample of any step.

Now only the middle sampling points at every step are used for the DFT. These samples of y (z too, similar hereinafter) are as shown below

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Fig. 1. A regular unequal-interval sampling, in which a step is divides six points, but only three points at middle are used to DFT.

$$y_{i,j}, 0 \leq i < n, s \leq j \leq t, 0 \leq s \leq \frac{m}{2}, \frac{m}{2} \leq t \leq m-1 \quad (4)$$

The Fig.1 expresses this model. The relative Fourier coefficients of the fundamental concerned can be proposed to calculate as:

$$a_1 = \frac{1}{t-s+1} \sum_{j=s}^t a_1(j), \quad a_1(j) = \frac{2}{n} \sum_{i=0}^{n-1} y_{i,j} \sin(i, j) \quad (5)$$

$$b_1 = \frac{1}{t-s+1} \sum_{j=s}^t b_1(j), \quad b_1(j) = \frac{2}{n} \sum_{i=0}^{n-1} y_{i,j} \cos(i, j) \quad (6)$$

Comparing with the well known general Fourier coefficients, some samples do not appear here, so that they are vacant at

some places. Obviously they do not belong to an equal-interval sampling. However it follows a certain rule, can be regarded as a regular unequal-interval sampling (RUEIS). It is a no studied project [4].

For convenience, it can be named the normal DFT used all point as all-point approach, and multi-point approach developed in this paper. For comparison a term of "single-point approach" is named in which only a point at middle is used.

B. Sine-wave case

Consider the n samples, $y_{0,s}, y_{1,s}, \dots, y_{n-1,s}$ (one sample comes from one step), which belong to a typical equal-interval sampling, and then the results are already known.

Afterwards, consider the second group of the n samples, namely, $y_{0,s+1}, y_{1,s+1}, \dots, y_{n-1,s+1}$, and up to the $[t-(s-1)]^{\text{th}}$ group, each group will have the same results, including same amplitude and same phase, due to the base functions moving their phase in the same way. Therefore, their average values of the Fourier coefficients will produce the same amplitude and phase, seen the appendix A. However noise influence is reduced.

C. Staircase waveform case

Similarly, consider the all $(t-s+1)$ group samples for the staircase waveform. In one group the values of the samples are different, indicating the height of the steps. Taking every group as a whole, it can be shown that all groups are the same. When the DFT is performed, all the groups will give out equal amplitude but different phases. The phase shift between the adjacent groups is equal to due to the start point of the base functions is displaced by an equal angle. According to the rule of vector-sum, the average amplitude of several groups will be less than that of single one.

When m , number of sample at one step, goes to infinity, and the total samples are used to calculate ($s=0, t=m-1$), the amplitude of fundamental will be equal to $\sin(h_1/2)/(h_1/2)$, which is the spectrum of the well known continuous-time staircase waveform [5]. The proof can be found in the appendix A.

III. PROPERTY OF STAIRCASE WAVEFORM

Although the staircase waveform will not directly be calculated separately in the differential measurement, its

Fig. 2 The sample distribution at the first step, against the step voltage measured by DVM

property in the RUEIS is still very important. For this aim some experiments are implemented. A commercial semiconductor DAC card with 16-bit resolution and an internal 10 V reference is used to produce a staircase waveform. Another commercial ADC card with a range of 2 V is used as a sampling device. The parameters of 24 bits@500 kS/s to 16 bits@ 15 MS/s can be available.

A. Distribution of Samples at one step

An experiment is done in which the DAC parameters are chosen as $n = 20$, the step function of $T = 5 \mu s$ to produce a 10 kHz staircase waveform. In order to observe clearly one step is divided as ten points, or $m = 10$. The sampling rate of $f_s = 2$ MS/s can satisfy this requirement. As the $\Sigma\Delta$ principle ADC's response for the staircase waveform appears significant fluctuation. An oversampling of $f_s = 10$ MS/s is adopted; its samples reflect the real situation of the staircase waveform [6].

The sample distribution of both cases at the first step is shown in Fig. 2, where for 10 MS/s the whole 50 samples at one step are divided 10 groups and the first one of every group is selected (similar hereinafter).

The data at every point in Fig. 2 are the differences against a reference. This reference is such obtained, a relative digit is imported to DAC, and the outputs in the DC model is measured by a high accuracy DC voltage meter (DVM, Agilent3458A, same hereinafter when the reference of the step voltage mentioned), which without any transient but with quantization.

It can be found that the transient at the start (and the end) is significant. There are about half points at the middle keeping a relative flat situation, the 10 MS/s appears more flat than the 2 MS/s. If the sign of $(s, t-s+1, m-1-t)$ is used to indicate the retained selection in (4), the (3,5,2) will be a good selection in Fig.2. However the small fluctuation exists still at the middle.

B. Fundamental Value for Different Retained Selections

In the same experiment the fundamentals of the staircase waveform are calculated according to (5, 6) (only for the samples selected the first one from every five samples). They are compared with the references of the step voltage. The results are shown in Table 1. Every result is through statistics from at least three measurements of the 5 periods each (similar hereinafter).

The results indicate that the larger amplitude appears in the case with the lesser points retained, agreeing with the analysis and that in (A7, A8). The all-point approach (0,10,0) is not a good choice due to transient; the multi-point approach (3,5,2) overcomes the transient reflected in (0,10,0); No significant variation between (4,4,2) and (3,5,2) means that it is reliable when select the about half samples. The single-point one overcomes transient too but with a bigger standard deviation.

TABLE I
FUNDAMENTAL VALUES COMPARISON

	(0,10,0)	(3,5,2)	(4,4,2)	(5,1,4)
Simulation in V	0.995932	0.999012	0.999382	0.999999
Experiment in V	0.995781	0.999019	0.999391	1.000003
Rel. Error in $\mu V/V$	-152.3	7.0	8.8	4.7
Std. Devi in $\mu V/V$	2.3	3.6	3.7	20.6

Fig. 3. A alignment of the differential sampling measurement between a sine-wave and a staircase waveform

IV. DIFFERENTIAL MEASUREMENT FOR SIN-WAVE

A. Alignment

As well known in the differential measurement the difference value of two signals is always expected as small as possible. In the single-point approach, if the under measured sine-wave likes the original one in Fig. 1, it must be moved its cross-zero point to the middle of the first step of the staircase waveform [1, 2], shown in Fig. 3.

$$y(x) = \sin(x - h_1/2) \quad (7)$$

Due to the signals of y of (7) and z of (2) have the same frequency and the amplitude; their curves will cross at similar phase point at every step. In this case the different values at all middle points are zero or very small.

In principle the multi-point approach takes the alignment relative easy due to more sampling points will be used.

In addition, increasing properly the values of n and m , the differences value of the two signals will become smaller. The analysis of the next section points that the adjustment of the m will be limited in non-sinusoidal wave measurement; in this case the n can freely be controlled to the asked level.

B. Experiment for a High-Purity Sine Wave

A differential measurement is undertaken, in which a high-purity sine-wave is produced by an AC voltage source (AVS, Fluke 5720) with amplitude about 1 V which is measured by an AC voltage standard meter (AVM, Fluke 5790) as a reference. A corresponding staircase waveform is provided by the DAC. The ADC measures their difference voltage with the suitable oversampling to reduce fluctuation of the ADC and ensure $m = 10$.

In addition the DAC gives out two synchronous signals, one to the ADC, and another to the AVS with a phase shift of $-h_1/2$ through parameter setting of DAC and the external precisely adjustment.

The fundamental of the sine-wave is calculated from the samples according to (5, 6). The digits of the staircase waveform are from (2). Its actual step voltages are measured in advance by a DVM. These step voltages are added to the

Fig. 4. Error of the amplitude of $c(j)$ against AVMS adopted the retained selection of (3,5,2), its distribution at the step

differential values sampled to form the precise new samples of the sine-wave for the following DFT. Several frequencies are tested with three approaches: the all-point, the multi-point, and the single-point.

The errors of the amplitude of $c_1(j) = \sqrt{a_1^2(j) + b_1^2(j)}$ against the AVMS, where $a_1(j)$, $b_1(j)$ defined in (5, 6) are shown in Fig. 4. The distribution at the ten points can be observed. The errors of the average value c_1 , the real amplitudes asked measuring are expressed in Table II for the different retained selections.

In the Fig. 4 a bigger transient effect exists at the starting three points of $c_1(0)$, $c_1(1)$, $c_1(2)$. The following five points are relative smooth. The variation ranges of the such five points are ± 10 , ± 70 , and ± 120 $\mu\text{V/V}$ for 1, 10, and 20 kHz respectively. It is just the retained selection of (3,5,2). When the frequency increases, the transient effect will significant enlarge.

In Table II, at the 1 kHz the errors of the three approaches are less than about 5 $\mu\text{V/V}$ closed to the AVMS, that means the system used is in a better situation, and the multi-point approach has no problem. In the case less than 1 kHz the wider step function of T sampled by the more points will take the transient no significant impact [7].

When frequencies are from 1 kHz to 20 kHz, the all-point approach of (0,10,0) displays its error increased 500 times, but the multi-point one of (3,5,2) only 10 times. Due to both cases have same channel and hence have similar hardware frequency response, their difference reflects the capability to overcome the transient effect.

In this frequency range the standard deviation of (3,5,2) changes 5 times whereas the single-point one of (5,1,4) closes to 10 times. In fact the multi-point approach could reach to a smaller standard deviation (hence, a lower uncertainty) if a higher sampling rate of ADC is applied and whole samples over the half part are retained (but not the 10 points limited).

TABLE II

SINE-WAVE MEASUREMENT: ERROR (STANDARD DEVIATION) IN $\mu\text{V/V}$				
f/kHz	n	(0,10,0)	(3,5,2)	(5,4,1)
1	20	2.6(1.3)	5.3(3.6)	5.6(8.2)
2	20	-0.7(8.1)	-5.8(11.9)	1.4(25.5)
5	20	-6.2(9.2)	-21.8(13.6)	-1.2(31.1)
10	20	167.0(6.6)	-17.4(15.6)	-73.6(32.3)
20	10	1021.9(13.4)	-53.8(17.3)	-84.9(74.1)
40	10	2839.8(15.1)	289.0(25.3)	118.7(96.1)

At the 40 kHz, its distribution is different than the others in Fig. 4, a bigger fluctuation exists in the distribution of $c_1(j)$, and only after $c_1(4)$ a relative flatness appears but within up to ± 500 $\mu\text{V/V}$. However a more or less capability to overcome the transient is found in Table II. It is worth to further investigate. The multi-point approach itself can in principle be applied to any frequency. But the system's feature, e.g. DAC's transient and ADC's response, will restrict the applied frequency. The highest frequency limit of the system used should be carefully investigated.

The experiment validates that the excellent capability of the multi-point approach to overcome the transient effect, and the potential capability to apply to higher frequency (single-tone).

V. ANALYSIS FOR HIGHER FREQUENCY COMPONENTS

A. A New matter

In the above differential measurement the high-purity sine-wave are asked. However this high-purity is not exactly known. It will introduce error in the some subsequent measurement, e.g. in the thermo-conversion. In this case it will not be enough to know the fundamental only. The multi-point approach provides an opportunity to check higher frequency components.

In fact the unequal-interval sampling will introduce a new matter when the higher frequency components are asked to analyze. A simple example can be used to explain it. Assuming that a staircase waveform has 10 steps, and every step is sampled with 6 points, in which only 3 points are selected as the retained part to DFT [$n = 10$, $m = 6$ and the retained selection is (2,3,1)]. If the highest frequency contained of the measured signal is less than $5f_0$, where f_0 is the fundamental frequency; the all components can be analyzed according to the Section IV. If the highest frequency is great than $30f_0$, (due to 60 points over one period) it is sure that this signal will not be analyzed. The rest problem here is, how about the frequencies between $5f_0$ to $30f_0$? Further the 30 points have already been discarded and only the 30 points retained, thus it will focus to a range of $5f_0$ to $15f_0$. This is a new matter for application of the Nyquist Theorem produced from the unequal-interval sampling.

Fig. 5. The Aliasing effect of ADC

When the highest frequency is less than $15f_0$, they can be calculated according to formula similar to (5, 6), as a known aliasing effect of ADC, shown in Fig. 5, for all $a_k(j)$, $b_k(j)$ the aliasing effect will take place at inside of the following groups: {1, 9, 11}, {2, 8, 12}, {3, 7, 13}, {4, 6, 14}, and {5, 15}. But it will not exist between any two groups. However the sums of $a_k(j)$, $b_k(j)$ will offer a new chance.

For example in the group of {1, 9, 11}, if the calculated coefficient is labeled as $\bar{C}_1 = (\bar{a}_1, \bar{a}_9, \bar{a}_{11}, \bar{b}_1, \bar{b}_9, \bar{b}_{11})$, which is always not so correct as the aliasing, and the original one is $C_1 = (a_1, a_9, a_{11}, b_1, b_9, b_{11})$, a relationship can be written as,

$$CC_1^T = \bar{C}_1^T \quad (8)$$

Where C is a 6×6 matrix. If its inverse matrix exists the original one can be calculated as following.

$$C_1^T = C^{-1} \bar{C}_1^T \quad (9)$$

In some cases, there is no mutual influence between the sine and cosine coefficient, so that the matrix C will become a simple form,

$$C = \begin{pmatrix} A & D \\ E & B \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \quad (10)$$

Where A and B are 3×3 matrix. For the group of {1, 9, 11}, {2, 8, 12}, {3, 7, 13}, {4, 6, 14}, the (10) can be replaced by simple forms as below. For $k = 1, 2, 3$, and 4,

$$\begin{pmatrix} a_k \\ a_{10-k} \\ a_{10+k} \end{pmatrix} = A_k^{-1} \begin{pmatrix} \bar{a}_k \\ \bar{a}_{10-k} \\ \bar{a}_{10+k} \end{pmatrix} \quad (11)$$

$$\begin{pmatrix} b_k \\ b_{10-k} \\ b_{10+k} \end{pmatrix} = B_k^{-1} \begin{pmatrix} \bar{b}_k \\ \bar{b}_{10-k} \\ \bar{b}_{10+k} \end{pmatrix} \quad (12)$$

The group of {5, 10, 15} is a different case, the cosine coefficient of the $\bar{b}_{15}$ is not able to analyze because it is proportional with the $\bar{b}_5$.

$$\begin{pmatrix} a_5 \\ a_{10} \\ a_{15} \end{pmatrix} = A_5^{-1} \begin{pmatrix} \bar{a}_5 \\ \bar{a}_{10} \\ \bar{a}_{15} \end{pmatrix} \quad (13)$$

$$\begin{pmatrix} b_5 \\ b_{10} \end{pmatrix} = B_5^{-1} \begin{pmatrix} \bar{b}_5 \\ \bar{b}_{10} \end{pmatrix} \quad (14)$$

The elements of the matrixes of A , B can be simulated. For example, for the retained selection of (2, 3, 1), they and their inverse forms are as following, and the matrix A_k (B_k) are same for $k = 1, 2, 3$, and 4.

$$A_k = \begin{pmatrix} 1 & 2/3 & -2/3 \\ 2/3 & 1 & 0 \\ -2/3 & 0 & 1 \end{pmatrix}, A_k^{-1} = \begin{pmatrix} 9 & -6 & 6 \\ -6 & 5 & -4 \\ 6 & -4 & 5 \end{pmatrix}$$

$$B_k = \begin{pmatrix} 1 & -2/3 & -2/3 \\ -2/3 & 1 & 0 \\ -2/3 & 0 & 1 \end{pmatrix}, B_k^{-1} = \begin{pmatrix} 9 & 6 & 6 \\ 6 & 5 & 4 \\ 6 & 4 & 5 \end{pmatrix}$$

$$A_5 = \begin{pmatrix} 5/3 & 0 & -2/3 \\ 0 & 1 & 0 \\ -2/3 & 0 & 2/3 \end{pmatrix}, A_5^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 5/2 \end{pmatrix}$$

$$B_5 = \begin{pmatrix} 1/3 & 0 \\ 0 & 1 \end{pmatrix}, B_5^{-1} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

The $10f_0$ is here expressed together with the $5f_0$ and the $15f_0$ for convenience. In fact the $10f_0$ is independent.

For the all-point approach of (0, 6, 0), the matrix of C becomes an unit matrix, $\bar{C}_1$ is just the actual amplitude of C_1 .

This is a well known fact.

The general formulas are analyzed in the Appendix B. Some interesting points are summarized as following.

For the three order matrix approach, the condition under which the simple form of (10) will exist is

$$t-s+1 = m/2, \quad t+s = m \quad (15)$$

For $k < n/2$,

$$A_k = \begin{pmatrix} 1 & \frac{2}{m} \operatorname{cosec} \frac{\pi}{m} & -\frac{2}{m} \operatorname{cosec} \frac{\pi}{m} \\ \frac{2}{m} \operatorname{cosec} \frac{\pi}{m} & 1 & 0 \\ -\frac{2}{m} \operatorname{cosec} \frac{\pi}{m} & 0 & 1 \end{pmatrix} \quad (16)$$

$$B_k = \begin{pmatrix} 1 & -\frac{2}{m} \operatorname{cosec} \frac{\pi}{m} & -\frac{2}{m} \operatorname{cosec} \frac{\pi}{m} \\ -\frac{2}{m} \operatorname{cosec} \frac{\pi}{m} & 1 & 0 \\ -\frac{2}{m} \operatorname{cosec} \frac{\pi}{m} & 0 & 1 \end{pmatrix} \quad (17)$$

For $k = n/2$,

$$A_{\frac{n}{2}} = \begin{pmatrix} 1 + \frac{2}{m} \operatorname{cosec} \frac{\pi}{m} & 0 & -\frac{2}{m} \operatorname{cosec} \frac{\pi}{m} \\ 0 & 1 & 0 \\ -\frac{2}{m} \operatorname{cosec} \frac{\pi}{m} & 0 & 1 - \frac{2}{m} \operatorname{cosec} \frac{\pi}{m} \end{pmatrix} \quad (18)$$

$$B_{\frac{n}{2}} = \begin{pmatrix} 1 - \frac{2}{m} \operatorname{cosec} \frac{\pi}{m} & 0 \\ 0 & 1 \end{pmatrix} \quad (19)$$

The combinations of m, t, s matched the conditions of (15) are partly listed in Table III. In fact the parameters of the single-point approach meet the rule expressed in the Table III. It means again that the single-point approach is a special one of

TABLE III
THE MATCHED CONDITIONS COMBINATIONS OF M, T, S

m	s	t	$t-s+1$
6	2	4	3
10	3	7	5
14	4	10	7

Fig. 6. Schematic diagram for differential sampling measurement between a non-sinusoidal wave and a staircase waveform

the general case.

The forms of the matrix of A and B (16-19) are depended on m only, will be unchanged for other n values.

It is proved that the inverse matrixes of A and B exist. That means the aliasing effect of RUEIS is solvable. The three order matrix approach is hence reasonable to extend the highest frequency component from $0.5n f_0$ to $1.5n f_0$.

This method has a potential application to measure non-sinusoidal wave based the differential measurement using of the staircase waveform as the reference.

B. Demonstrative Experiment for Non-Sinusoidal Wave

A simulation experiment is undertaken for a differential sampling measurement of a non-sinusoidal wave with several higher frequency components when a relative staircase waveform is taken as reference. The 60 samples over one period and in only 30 samples are retained with (2,3,1). The above proposed method is applied. The solved components are agreed to the original setting one without significant errors.

The second experiment is implemented to validate the proposed method. A real non-sinusoidal wave produced by a signal generator (Source, Agilent33522A) is sampled by the ADC in full scale resulted in the 60 samples over one period. The same samples are analyzed from the fundamental to the 15th harmonic by two approaches respectively: the normal DFT, i.e. (0,6,0) and the multi-point one of (2,3,1). The both results are agreed within the level of 2×10^{-5} V when the maximum amplitude of signal less than 1 V.

An actual differential sampling experiment is done in a proposed scheme shown in Fig. 6. A non-sinusoidal wave is produced by the Source with fundamental and the 3rd, 5th and 11th harmonics with the maximum amplitude less than 1 V.

A novel alignment is adopted to meet the requirement. The non-sinusoidal wave is sampled by the ADC with 60 samples in one period. These 60 samples are divided into the 10 groups of the 6 each. The step voltage of the needed staircase waveform is determined through selection such sample which index is $j=3$, or $z_i = y_{i,3}$. Thus the differences at these points will become zero. It will ensure that the retained selection of (2,3,1) and the relative multi-point approach can be implemented.

According to these step voltages the DAC produces then the

Fig. 7. A scheme for differential sampling of a non-sinusoidal wave

staircase waveform. The both signals are differential sampled by the ADC, their waveforms shown in Fig. 7. It can be noticed that a better alignment is realized.

The step voltages produced by DAC in their DC model when importing the determined values aforementioned are measured by DVM one by one. The sum of these step voltages and the samples (differences) are calculated through DFT with the multi-point approach, and then through the calculation of $A^{-1} B^{-1}$.

The experiment is repeated for 50 signal's periods. The obtained results are compared with that of the normal DFT of the (0,6,0) to the full scale sampling. The agreement is within 3×10^{-5} V. It is agreed with the experimental standard deviation (several 10^{-5} V but less than 1×10^{-4} V) which reflects mainly the stability of the Source, and of course includes the level of synchronous control.

In the differential sampling the step voltages are main part which can easy be measured by DVM at a higher level. The differential part can be controlled less than 1×10^{-2} V or lower through the proposed alignment. Thus a general device of ADC will offer a better result.

Whatever the experiments validate that the aliasing effect due to discard some samples has already been solved. The 30 retained samples in the RNEIS situation can be analyzed up to the highest frequency of the 15 time fundamental. And an usable scheme for non-sinusoidal differential measurement is proposed.

VI. CONCLUSION

A multi-point approach to overcome transient effect is proposed. The fundamental amplitude of the staircase waveform is experimentally compared for different retained selections. The case when all points are retained is not a good selection due to transient. Whereas single-point one is with a significant standard deviation. The experiment for a sine-wave demonstrates the advantage of the multi-point approach in lower error, smaller standard deviation, and wider applying range of frequency. A new method is developed to fit the regular unequal-interval sampling to extend application of Nyquist Theorem. The three order matrix approach can extend the highest frequency component of the measured signal from

0.5 πf_0 to 1.5 πf_0 . The elements of the matrix are given out in digital as well as in formula. They are unchanged for the different step number n of the staircase waveform. The experiments for a non-sinusoidal wave validate the capability, and offer a differential measurement scheme.

APPENDIX

A. Fundamental Calculation

For the sine signal (1), we can calculate as below [8].

$$a_1(j) = \frac{2}{n} \sum_{i=0}^{n-1} \sin(i, j) \sin(i, j) = \frac{2}{n} \sum_{i=0}^{n-1} \sin^2(imh_2 + jh_2) \\ = 1 - \frac{1}{n} [\cos(2jh_2 + (n-1)mh_2) \sin(nmh_2) \operatorname{cosec}(mh_2)] \quad (A1) \\ = 1$$

$$b_1(j) = \frac{2}{n} \sum_{i=0}^{n-1} \sin(i, j) \cos(i, j) = \frac{1}{n} \sum_{i=0}^{n-1} \sin 2(imh_2 + jh_2) \\ = \frac{1}{n} [\sin(2jh_2 + (n-1)mh_2) \sin(nmh_2) \operatorname{cosec}(mh_2)] \quad (A2) \\ = 0$$

That means all $a_1(j)$ is equal to 1, and all $b_1(j)$ to 0. Thus, $a_1 = 1$ $b_1 = 0$.

As for the staircase waveform (2), the calculation becomes,

$$a_1(j) = \frac{2}{n} \sum_{i=0}^{n-1} \sin(i, j) \sin(i, j) = \frac{2}{n} \sin(i, 0) \sin(i, j) \\ = \frac{2}{n} \sum_{i=0}^{n-1} \sin(imh_2) \sin(imh_2 + jh_2) = \frac{1}{n} \{-\cos[jh_2 + \\ + (n-1)mh_2] \sin(nmh_2) \operatorname{cosec}(mh_2) + \cos(-jh_2)\} \\ = \cos(-jh_2) \quad (A3)$$

$$b_1(j) = \frac{2}{n} \sum_{i=0}^{n-1} \sin(i, j) \cos(i, j) = \frac{2}{n} \sin(i, 0) \cos(i, j) \\ = \frac{2}{n} \sum_{i=0}^{n-1} \sin(imh_2) \cos(imh_2 + jh_2) = \frac{1}{n} \{-\sin[jh_2 + \\ + (n-1)mh_2] \sin(nmh_2) \operatorname{cosec}(mh_2) + \sin(-jh_2)\} \\ = \sin(-jh_2) \quad (A4)$$

The average among the above results from s th to t th can be obtained as

$$a_1 = \frac{1}{t-s+1} \sum_{j=s}^t \cos(-jh_2) = \frac{1}{t-s+1} \sum_{k=0}^{t-s} \cos(-sh_2 - kh_2) \\ = \frac{1}{t-s+1} \{\cos[(s+t)(-h_2/2)] \sin[(t-s+1)(-h_2/2)] \cdot \\ \cdot \operatorname{cosec}(-h_2/2)\} \quad (A5)$$

$$b_1 = \frac{1}{t-s+1} \sum_{j=s}^t \sin(-jh_2) = \frac{1}{t-s+1} \sum_{k=0}^{t-s} \sin(-sh_2 - kh_2) \\ = \frac{1}{t-s+1} \{\sin[(s+t)(-h_2/2)] \sin[(t-s+1)(-h_2/2)] \cdot \\ \cdot \operatorname{cosec}(-h_2/2)\} \quad (A6)$$

The amplitude and the phase angle become hence:

$$c_1 = \frac{1}{t-s+1} \sin[(t-s+1)(-h_2/2)] \operatorname{cosec}(-h_2/2) \quad (A7)$$

$$\varphi_1 = (s+t)(-h_2/2) \quad (A8)$$

When all samples in the step are used to calculation, or, $s = 0$, and $t = m-1$, the results become,

$$c_1 = \sin(-\pi/n) \operatorname{cosec}(-\pi/nm) \quad (A9)$$

$$\varphi_1 = -\pi(m-1)/(nm) \quad (A10)$$

Further, when $m \rightarrow \infty$

$$c_1 = \sin(\pi/n)/(\pi/n) \quad (A11)$$

$$\varphi_1 = (-\pi/n) \quad (A12)$$

B. Inverse Matrix for higher frequency components

The following discussion focuses on the three order matrix approach. The representative elements are proved as following only.

1) For case of $k < n/2$

In the condition of a non-sinusoidal signal, for any $l = k, n-k, n+k$, the elements of the matrix A can be calculated, according to the definition of (5, 6)

$$a_{l,l}(j) = \frac{2}{n} \sum_{i=0}^{n-1} \sin l(imh_2 + jh_2) \sin l(imh_2 + jh_2) \\ = \frac{2}{n} \sum_{i=0}^{n-1} \frac{1}{2} [1 - \cos(2l imh_2 + j2lh_2)] \\ = \frac{1}{n} \{n - \cos[(j2lh_2 + (n-1)lmh_2) \sin nlmh_2 \operatorname{cosec} l mh_2] \\ = 1$$

Where relations of $\sin nlmh_2 = \sin l2\pi = 0$ and $\operatorname{cosec} l mh_2 \neq \operatorname{cosec} \frac{1}{2} nmh_2 = \operatorname{cosec} \pi = \infty$ are applied.

$$a_{l,l} = \frac{1}{(t-s+1)} \sum_{j=s}^t 1 = 1 \quad (B1)$$

$$a_{k,n-k}(j) = \frac{2}{n} \sum_{i=0}^{n-1} \sin k(imh_2 + jh_2) \sin(n-k)(imh_2 + jh_2) \\ = \frac{-1}{n} \sum_{i=0}^{n-1} [\cos n(imh_2 + jh_2) - \cos(n-2k)(imh_2 + jh_2)] \\ = \frac{-1}{n} \sum_{i=0}^{n-1} \{\cos jnh_2 - \cos[2kmh_2 - j(n-2k)h_2]\} \\ = \frac{-1}{n} \{n \cos jnh_2 - \cos[j(2k-n)h_2 + \\ + (n-1)kmh_2] \sin nkmh_2 \operatorname{cosec} kmh_2\} \\ = -\cos jnh_2$$

Thus,

$$\begin{aligned}
a_{k,n-k} &= \frac{-1}{t-s+1} \sum_{j=0}^{t-s} \cos(sn h_2 + j n h_2) \\
&= \frac{-1}{t-s+1} \cos(sn h_2 + \frac{t-s}{2} n h_2) \sin \frac{t-s+1}{2} n h_2 \operatorname{cosec} \frac{n h_2}{2} \\
&= \frac{-1}{t-s+1} \cos(\frac{t+s}{2} n h_2) \sin \frac{t-s+1}{2} n h_2 \operatorname{cosec} \frac{n h_2}{2}
\end{aligned}$$

When $t-s+1 = m/2$, $t+s = m$,

$$a_{k,n-k} = a_{n-k,k} = \frac{2}{m} \operatorname{cosec} \frac{\pi}{m} \quad (\text{B2})$$

As for the matrix of **D**, **E** in (10),

$$\begin{aligned}
d_{i,j}(j) &= \frac{2}{n} \sum_{i=0}^{n-1} \sin l(im h_2 + j h_2) \cos l(im h_2 + j h_2) \\
&= \frac{1}{n} \sum_{i=0}^{n-1} \sin(i2l m h_2 + j2l h_2)
\end{aligned}$$

$$\begin{aligned}
&= \sin(j2l h_2 + (n-1)l m h_2) \sin l n m h_2 \operatorname{cosec} l m h_2 \\
&= 0
\end{aligned}$$

Thus

$$d_{i,j} = 0 \quad (\text{B3})$$

$$d_{k,n-k}(j) = \frac{2}{n} \sum_{i=0}^{n-1} \sin k(im h_2 + j h_2) \cos(n-k)(im h_2 + j h_2)$$

$$= \frac{1}{n} \sum_{i=0}^{n-1} [\sin n(im h_2 + j h_2) + \sin(2k-n)(im h_2 + j h_2)]$$

$$= \frac{1}{n} \sum_{i=0}^{n-1} \{\sin j n h_2 + \sin[i2k m h_2 + j(2k-n)h_2]\}$$

$$= \frac{1}{n} \{n \sin j n h_2 +$$

$$+ \sin[j(2k-n)h_2 + (n-1)k m h_2] \sin k n m h_2 \operatorname{cosec} k m h_2\}$$

$$= \sin j n h_2$$

$$d_{k,n-k} = \frac{1}{t-s+1} \sum_{j=0}^{t-s} \sin(sn h_2 + j n h_2)$$

$$= \frac{1}{t-s+1} \sin(\frac{t+s}{2} n h_2) \sin \frac{t-s+1}{2} n h_2 \operatorname{cosec} \frac{n h_2}{2}$$

When $t-s+1 = m/2$, $t+s = m$,

$$d_{k,n-k} = 0. \quad (\text{B4})$$

2) Other relationships

From the condition (14), we obtain that

$s = (m+2)/4$, $t = (3m-2)/4$. The possible solutions are listed in Table III.

Under the conditions, all elements expressed in (15-19) are depended only on the parameter of m . That means these element formula can be applied to other n .

The determinant of the matrix **A** and **B** can be checked.

$$\begin{aligned}
|A| &= \begin{vmatrix} 1 & \frac{2}{m} \operatorname{cosec} \frac{\pi}{m} & -\frac{2}{m} \operatorname{cosec} \frac{\pi}{m} \\ \frac{2}{m} \operatorname{cosec} \frac{\pi}{m} & 1 & 0 \\ -\frac{2}{m} \operatorname{cosec} \frac{\pi}{m} & 0 & 1 \end{vmatrix} \\
&= 1 - 2(\frac{2}{m} \operatorname{cosec} \frac{\pi}{m})^2 = |B|
\end{aligned} \quad (\text{B5})$$

When $m = 4$, it is equal to 0. The other integers in the interval of $-100 < m < 100$, specially the integers listed in Table III, will make this determinant to no zero. That means the inverse matrix of **A** and **B** will exist. It points out a very important fact: the aliasing effect of RUEIS is solvable in the conditions we discussed. The Nyquist Theorem is hence still correct.

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