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Benchmark Studies for the Generalized Streamwise Periodic Heat Transfer Problem

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This is a comparison of calculations performed with a scheme for handling streamwise-periodic boundary conditions with known solutions to the common problem of fully developed heat transfer in a plane duct. Constant value, constant flux, mixed boundary conditions, and linear wall flux (conjugate heat transfer) are all considered. Agreement is, in every case, near exact showing that the methodology may be applied with confidence to complex engineering problems with a variety of thermal wall boundary conditions. [DOI: 10.1115/1.2955477]

Keywords: computational fluid dynamics, periodic boundary conditions, heat transfer, plane duct

Introduction

Computational fluid dynamics has been used on occasion to improve the design of heat exchangers. Frequently, streamwise-periodic boundary conditions are employed to perform calculations for fully developed flow and heat transfer in situations where a single unit cell represents a geometry repeated numerous times in the flow direction over some characteristic length scale, l . Figure 1(a) illustrates one such example. Under the circumstances, suitably reduced or nondimensionalized pressure and temperature fields, asymptotically approach constant values. Patankar et al. [1] pioneered the development of methodologies for periodic heat transfer under conditions of either constant heat flux (Neumann) or constant wall temperature (Dirichlet) boundary conditions. These approaches were subsequently applied by numerous other researchers in numerical heat transfer work. The reader will note that the choice of state variable in Ref. [1] differs for the constant T_w and q'' boundary value problems. For this reason the approach cannot readily be adopted for problems involving combinations of Neumann and Dirichlet boundary conditions, or for the intermediate linear (Robin) boundary value problem. Mixed boundary conditions arise naturally, e.g., within the passages of heat exchangers transferring energy between two working fluids.

Recently the present author [2] proposed a rationale, which could, in principle, be applied not only to the Dirichlet and Neumann problems, but also to the intermediate linear boundary value problem, as well as combinations of different wall boundary conditions. This is possible because primitive variables, temperature and pressure, were selected as state variables. However, the only example given in Ref. [2] was a 3D constant T_w offset-fin heat exchanger geometry. The purpose of this article is to present comparisons of the methodology [2] for a simple geometric case, namely, the fully developed Graetz problem, over a range of thermal boundary conditions, and to demonstrate that it is possible to obtain fully developed solutions under regularly repeating but ar-

bitrary wall boundary conditions using a primitive-variable formulation, i.e., to validate the methodology. The deliberate choice of simplified geometry allows for exact comparison with data from the standard literature to be made without compromise in terms of the general nature of the methodology and its potential for practical application.

Periodic Boundary Conditions

The general details of how periodic boundary conditions are applied to the flow by means of a pressure jump or shock, were provided in Ref. [2]. Therefore only material related to heat transfer is provided below. The streamwise-periodic thermal condition may be expressed [3] as follows:

$$T(0) = c_1 T(l) + c_2 \quad (1)$$

with

$$c_1 = \frac{T_0(0) - T_w(0)}{T_0(l) - T_w(l)} \quad (2)$$

$$c_2 = T_0(0) - c_1 T_0(l) \quad (3)$$

where T_0 is a suitable reference, e.g., local bulk temperature. Thus, the temperature difference or jump between the given upstream and downstream values is $\Delta T = (c_1 - 1)T + c_2$, an expression, which will prove useful when prescribing periodic boundary conditions below. For $T_w \neq \text{constant}$, the upstream wall values must be computed from those downstream

$$T_w(0) = \frac{T_0(0) + B(l)T_\infty}{1 + B(l)} \quad (4)$$

where the driving force [4] is

$$B = \frac{T_0 - T_w}{T_w - T_\infty} \quad (5)$$

and T_∞ is an ambient or external value. Periodicity is defined here as being the case if and only if the local heat transfer driving force is cyclic,

$$B(x + l, y) = B(x, y) \quad (6)$$

It being assumed that the upstream bulk value, $T_0(0)$, is prescribed a priori, whereas the downstream value $T_0(l)$ is computed iteratively at run time.

Benchmark Cases

The methodology was tested by obtaining performance calculations for fully developed flow and heat transfer in ducts, for which solutions are known for a variety of wall conditions. Table 1 shows the four cases considered.

- (1) constant wall temperatures, T_1 and T_2 (Dirichlet)
- (2) constant heat fluxes q_1'' and q_2'' (Neumann)
- (3) one constant heat flux, q_1'' , and one constant temperature, T_2 (mixed)
- (4) heat flux prescribed according to a linear rate equation (Robin), namely,

$$q'' = U(T_\infty - T_w) \quad (7)$$

a physical interpretation being the transfer of heat to walls of overall heat transfer coefficients, U_1 and U_2 , which in turn transfer heat to two reservoirs, at temperatures $T_{1\infty}$ and $T_{2\infty}$. Under the circumstance, a nondimensional wall resistance [5] is defined, for each wall, by

$$R_w = \frac{k}{UD_h} \quad (8)$$

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Fig. 1 Example of streamwise-periodic geometry showing unit cell geometry for (a) offset fin and (b) plane duct showing the mesh and boundary conditions

This is related to the Biot number according to $R_w = 1/(4Bi)$. Cases (1)–(4) in Table 1 summarize the performance of these four commonly encountered internal forced convection problems [6] in terms of the Nusselt number.

Implementation

In computational heat transfer codes based on the finite-volume method [7], a set of linear algebraic equations having the form

$$\sum a_{NB}(T_{NB} - T_P) + S = 0 \quad (9)$$

is solved for steady flow. T_P is the in-cell value of T at any given computational cell, with neighbors NB=W (west), E (east), S (south), and N (north). NB: With temperature, T , as state variable, the units of the linking neighbor coefficients, a_{NB} , are W/K as opposed to the more usual units of kg/s when enthalpy is dependent variable.

The wall boundary conditions in Eq. (9) are linearized according to

$$S = C(T_\infty - T_P) \quad (10)$$

where C is a source-term coefficient and T_∞ is a source-term value. Equation (10) follows the notation of Spalding [8], which is consistent with the zero-flux form of Eq. (9). Patankar [7] employed the form $S = S_C + S_P T_P$ with $S_C = CT_\infty$ and $S_P = -C$.

For Case (4), the source-term coefficient and wall value are computed using harmonic and arithmetic averaging [7],

$$\frac{1}{C} = \frac{\delta y}{kA_N} + \frac{1}{UA_N} \quad (11)$$

$$T_w = \frac{k/\delta y}{k/\delta y + U} T_P + \frac{U}{k/\delta y + U} T_\infty \quad (12)$$

As noted in Ref. [9], the Dirichlet and Neumann cases are easily constructed as limits of the linear Case (4). This is achieved by (i) setting the resistance, R_w , to a very small number, so that U and C are correspondingly large (Dirichlet), or (ii) by setting both R_w and $T_\infty = \dot{q}''/C$ to large numbers (Neumann).

Table 1 Benchmark problems and results

			Previous work		Present study			
			Nu ₁	Nu ₂	Nu ₁	% error	Nu ₂	% error
1.		(a) $T_1 = T_2$	7.541	7.541	7.536	0.066	7.536	0.066
		(b) $T_1 \neq T_2$	4.000	4.000	4.000	0	4.000*	—
2.		(a) $\dot{q}_1'' = \dot{q}_2''$	8.235	8.235	8.232	0.036	8.232	0.036
		(b) $\dot{q}_1'' = 0$	0.000	5.384	0	0	5.384	—
		(c) $\dot{q}_1'' \neq \dot{q}_2''$	Fig. 3		Fig. 3			
3.		(a) $\dot{q}_1'' = 0$	0.000	4.861	0.000	0	4.860	0.020
		(b) $\dot{q}_1'' \neq 0$	4.000	4.000	4.000	0	4.000*	—
4.		(a) $T_{1\infty} = T_{2\infty}$ $U_1 = U_2$	Fig. 4		Fig. 4			
		(b) $T_{1\infty} \neq T_{2\infty}$	4.000	4.000	4.001	0.025	4.001*	0.025

*For these cases, the temperature field is cyclic rather than periodic, hence $c_1 = 1$, $c_2 = 0$, and $T_0(0)$ is not prescribed.

Fig. 2 Boundary condition implementation showing (a) jump boundary condition at the inlet/outlet and (b) wall boundary notation

The streamwise-periodic boundary condition is prescribed [2] in two steps: (i) By activating true cyclic boundary conditions across the inlet/outlet in the streamwise direction, storage is assured for the missing streamwise linking coefficients, a_{NB} , in the energy and momentum equations at what would otherwise be the inlet and outlet of the domain, creating an endless flow loop. (ii) At the inlet a step or jump in the temperature field across the boundary is imposed by prescribing a line of source terms $S = a_E[(c_1 - 1)T_E + c_2]$ in computational cells immediately downstream of the periodic boundary, as shown in Fig. 2(a). Combining these with the terms $a_W(T_W - T_P)$ in Eq. (9) has the effect of replacing T_W with $c_1 T_W + c_2$, from Eq. (1). For parabolic situations, such as considered here, no further considerations are required.

For elliptic problems, Eq. (1) may be rewritten in the form

$$T(l) = c_1' T(0) + c_2' \quad (13)$$

and a similar line of source terms $S = a_E[(c_1' - 1)T_W + c_2']$ must be prescribed immediately upstream of the boundary at the outlet [10], as shown in Fig. 2. For the particular case of constant T_w , the present formulation is equivalent to that of Kelkar and Patankar [11]; however, it is more generally used for a wide range of thermal boundary conditions.

A staggered velocity scheme with hybrid differencing of convection and diffusion terms [7] was implemented in the PHOENICS code [12], which is based on a version of the SIMPLE algorithm [7,8,13]. Calculations were performed with meshes of both 2×100 and 10×100 cells in the x - y plane, similar to that shown Fig. 1(b), and concentrated by means of a geometric progression toward the walls where velocity and temperature gradients are maximum. (NB: A minimum of two cells in the x -direction is required for the present method.) Regardless of the number of cells in the x -direction, the values of local Nusselt numbers, Nu_1 and Nu_2 , at the north and south walls, were observed to be invariant. These were obtained as follows:

$$Nu = \frac{D_h}{k} C \left(\frac{T_P - T_\infty}{T_0 - T_w} \right) \quad (14)$$

where T_P is the in-cell value of T nearest the appropriate north or south wall, Fig. 2(b), and C is the source-term coefficient, from Eq. (10).

Fig. 3 Comparison of the present work with the expression of Shah and London [6] for Case 2(c)

Results

Table 1 provides a comparison of the present results, in terms of Nusselt numbers, with those of Shah and London [6] and others. These are discussed briefly below.

(1) *Dirichlet problem.* Agreement with analytical solutions is within 0.07% for (a) $T_1 = T_2$ and (b) $T_1 \neq T_2$.

(2) *Neumann problem.* (a) $\dot{q}_1'' = \dot{q}_2''$ agreement is within 0.04%. (b) $\dot{q}_1'' = 0$ agreement is exact. (c) $\dot{q}_1'' \neq \dot{q}_2''$. Figure 3 shows that agreement is excellent for this case. For $\dot{q}_2'' = \dot{q}_1'' \leq 2.5$ the present data agree with Ref. [6] to within 0.3%, and substantially better as $\dot{q}_2'' = \dot{q}_1'' \rightarrow 0$. Note that as $\dot{q}_2'' = \dot{q}_1'' \rightarrow 26/9$, $Nu \rightarrow \infty$ because $T_1 \rightarrow T_0$.

(3) *Mixed problem.* (a) $\dot{q}_1'' = 0$ agreement is within 0.02%. (b) $\dot{q}_1'' \neq 0$ exact agreement is obtained.

(4) *Linear boundary values.* (a) $T_{1\infty} = T_{2\infty}$. Figure 4 is a comparison of the present results with those of Sideman et al. [5] in terms of Nusselt number as a function of wall resistance. Agreement is always within 0.07%. (b) $T_{1\infty} \neq T_{2\infty}$. The same result is obtained as for Cases 1(a) and 3(b), regardless of whether $U_1 = U_2$ or not, showing these to be special cases of 4(b).

Discussion and Conclusion

Comparison of the present numerical work with previously obtained analytical and numerical results of others indicates that it is possible to prescribe fully developed streamwise-periodic boundary conditions under a range of thermal conditions. The results show that multiple wall temperatures/fluxes, a combination of constant temperature and constant flux, and linear wall conditions

Fig. 4 Comparison of the present work with the data of Sideman et al. [5] for Case 4(a)

can all be handled by the methodology. Although only pairs of boundary values were considered for a plane duct, the extension to a manifold of boundary fluxes and temperatures, and/or complex geometries and recirculating flow regimes with or without mass transfer at the walls, is readily apparent. Performance calculations for such problems are generally correlated with empirical, rather than analytical data.

Cases 1(a), 2(a)–(c), 3(a), and 4(a) all admit fully developed solutions where the solution is independent of the inlet bulk temperature and heat is transferred to or from the working fluid. Cases 1(b), 3(b), and 4(b) only allow for cyclic solutions where no net heat is transferred to or from the working fluid, and the bulk temperature is simply a weighted average of wall values. It is important, when considering more complex geometries, to identify which class of problem is under consideration. For example, for crossflow in a bank of finned tubes, there may be temperature variations from fin tip to root, and a periodic solution still achieved; on the other hand, if alternate rows are at different temperatures, only a cyclic solution is possible. Nusselt number correlations for the former class of problem are, generally speaking, of more interest to the heat exchanger designer. It is also important to appreciate that not every problem involving regularly repeating geometry will necessarily admit a steady fully developed periodic solution [14].

In view of the wide range of Nusselt numbers observed in Figs. 3 and 4, the presumption of constant flux or constant temperature formulations for engineering problems will lead to substantial errors, when applied to periodic problems where these are, in fact, variable, and is neither recommended nor necessary. A conclusion of this study is that the primitive-variable streamwise-periodic formulation generates reliable results over a range of thermal boundary conditions, as verified by analytical and other data. Based on this observation, the methodology may be applied with confidence to performance calculations for complex periodic heat and mass transfer problems.

Nomenclature

a	= linking coefficient, W/K
A_N	= area of north face of computational cell, m ²
C	= source-term coefficient, W/K
D_h	= hydraulic diameter, m
k	= thermal conductivity, W/(m K)
l	= length, m
T	= temperature, K
$\dot{q}''$	= heat flux, W/m ²
S	= source term, W

U = overall heat transfer coefficient, W/(m²K)

δy = cell half distance, m

Nondimensional Numbers

B = heat transfer driving force

Bi = Biot number

Nu = Nusselt number

R_w = thermal resistance

Subscripts

E, W, S, N = east, west, south, and north neighbors

NB = neighbor

P = nodal

w = wall

0 = reference, bulk

∞ = external, ambient

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