NRC Publications Archive
Archives des publications du CNRC

Representation of 3-D and 4-D Objects based on an Associated Curved Space and a General Coordinate Transformations Invariant Description
Paquet, Eric

National Research Council Canada   Conseil national de recherches Canada

Canada

National Research
Council Canada

Conseil national
de recherches Canada

Institute for
Information Technology

Institut de technologie
de l'information

NRC·CNRC

*Representation of 3-D and 4-D Objects based
on an associated Curved Space and a
General Coordinate Transformation
Invariant Description ***

Paquet, E.
2006

Canada

Representation of 3-D and 4-D Objects based on an Associated Curved Space and a General Coordinate Transformations Invariant Description

Eric Paquet

Visual Information Technology, National Research Council, M-50 Montreal Road
Ottawa (Ontario) K1A 0R6 Canada
eric.paquet@nrc-cnrc.gc.ca

Abstract. This paper presents a new theoretical approach for the description of multi-dimensional objects for which 3-D and 4-D are particular cases. The approach is based on a curved space which is associated to each object. This curved space is characterised by Riemannian tensors from which invariant quantities are defined. A descriptor or index is constructed from those invariants for which a statistical and an abstract graph representation are associated. The obtained representations are invariant under general coordinate transformations. The statistical representation allows a compact description of the object while the abstract graph allows describing the relations in between the parts as well as the structure.

1 Introduction

Content-based description plays a prominent role in indexing and retrieval [1-4]. It is therefore important to develop invariant representations for 3-D objects. An excellent review about indexing and retrieval of 3-D objects can be found in [1-3]. As can be seen from this review, most of the proposed techniques are invariant under a very limited class of transformations, e.g. translations, scaling and rotations. Relatively less attention has been devoted to the development of representations that are invariant under *general coordinate transformations*. In addition, most approaches are limited to 3-D objects understood in the sense of 2-D surfaces embedded in 3-D space (e.g. a VRML object) and cannot be applied to volumetric objects, like those generated by tomography. Such multidimensional objects are characterised by the fact that each point in 3-D space (volumetric
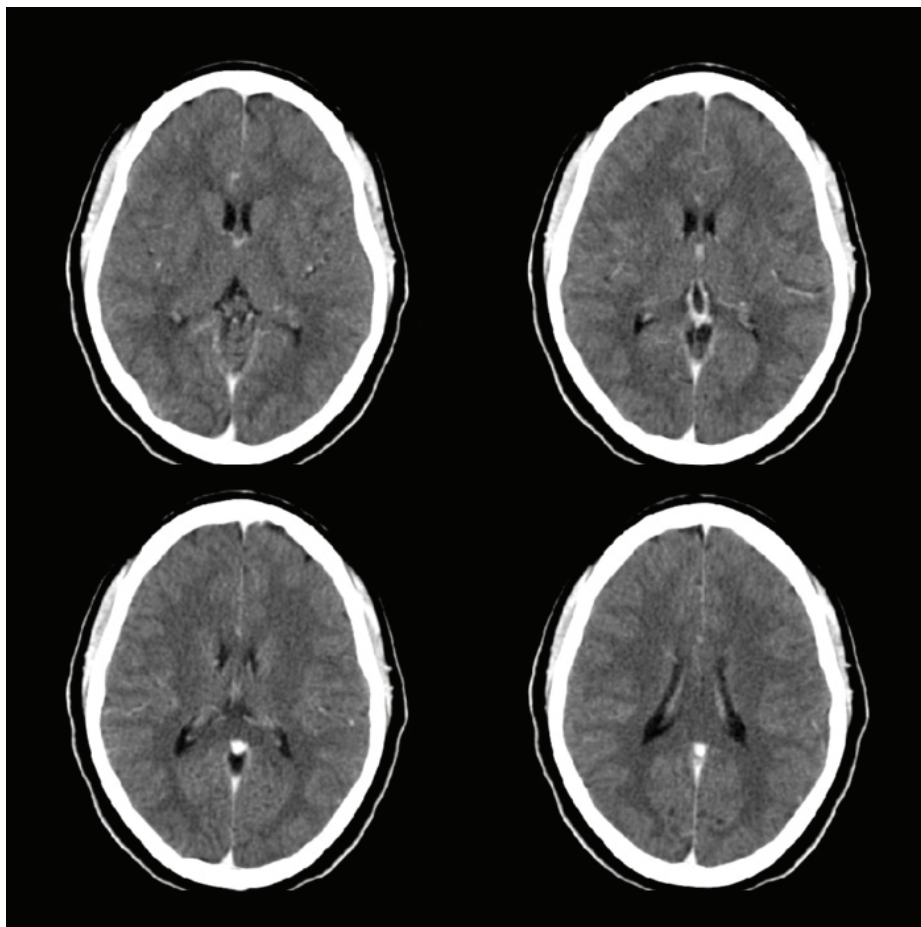
space) is associated with a set of attributes. For instance, in the case of tomography, the set is generally limited to one attribute, the density, and the map is called a 4-D object. This paper presents a new approach for invariant description of multi-dimensional objects under general coordinate transformations leading to a new type of representation based on the Ricci tensor and scalar. This novel approach transposes certain results of general relativity [5-7] and Riemannian geometry [5] into the framework of computer vision.

The paper is organised as follows. After some considerations on content-based indexing and retrieval, we review the most important results of tensor analysis which are necessary to understand our approach. Then, a tensor is associated to each object and the fundamental equations are derived from a variational principle. This tensor describes the attributes of the object and becomes the source for an associated curved space. The geometry of this associated curved space is described by a quadric form or a metric, a Ricci tensor and a Ricci scalar from which invariant quantities are derived. Finally, two representations are adopted for the invariants. The first one is a statistical representation based on a novel histogram and the second representation is a topological one based on an abstract graph. These representations are invariant under general coordinate transformations.

2   Content-based Description of 3-D and 4-D Objects

An important challenge in content-based description is to find a representation which is invariant under arbitrary coordinate transformations. Furthermore, such a description should not be limited to 3-D objects, but should be easily extendable to multi-dimensional object (such as 4-D tomography, as illustrated in Figure 1). The extension to multidimensional content is problematic due to the high number of dimensions involved, their heterogeneity (space, time, speed, density, field intensity, etc.) and the fact that the standard mathematical framework has proven itself to be not suitable to derive form invariant equations under arbitrary coordinate transformations [5-7]. Form invariance is important in order to construct a description that is invariant under arbitrary coordinate transformations. This means that, no matter how the initial object is transformed (as defined in section 3), the associated description is always the same. A new

approach is presented to this paper, based on tensor analysis, in which a Riemannian space or curved space (as opposed to standard flat Euclidian space) is associated to an object. This space is described by tensorial equations which are form invariant under arbitrary coordinate transformations. A set of invariant quantities are extracted from this space and a representation is constructed from the set of invariants. Two types of representation are considered: Riemannian histograms and abstract graphs.

Figure 1. Four views of a tomographic image of the brain. The intensity is related to the density. Each slice corresponds to a certain elevation in the brain.

3 An Overview of Tensor Analysis

In this section, an overview of tensor algebra is presented. Derivations and more details can be found in [5-8]. This section is a prerequisite for what follows. Unless stated otherwise, all Greek indices and all summations are to be taken from 1 to N. Furthermore, if an index is not involved in a summation, it is immaterial and can be replaced by any other index.

The use of tensorial analysis is justified by the fact that tensorial equations do not change their form under arbitrary coordinate transformations. We assume that the space has an arbitrary number of dimensions, which means that the equations that we will derived could be applied to both 3-D and 4-D objects. A point in space is given by

$$(1) \qquad\qquad x = \left\{ x^\mu \right\}$$

For instance in 3-D

$$(2) \qquad\qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

We assume that it is possible to associate to the space a metric $g_{\mu\nu}(x)$, which is defined by the quadratic form

$$(3) \qquad\qquad ds^2 \equiv \sum_{\mu\nu} g_{\mu\nu}(x) dx^\mu dx^\nu$$

In other words, we assume that we can define an invariant distance *locally*. This is an important distinction in between standard Euclidian geometry and Riemannian geometry: distance is a *global* invariant (for orthogonal transformations) for the former and a local invariant for the latter (under general coordinate transformations). Indeed, because of the *space curvature*, it is not possible to define a global metric. It should be noticed that ds, the infinitesimal length of arc, is an invariant and has the same value irrespectively of the coordinate system. Note that this metric defines the inner product between two tensors for the curved space.

The covariant and a contravariant vector are defined respectively as

(4) $\quad A'_\mu(x') = \sum_\nu \dfrac{\partial x^\nu(x')}{\partial x'^\mu} A_\nu(x)$ $\quad$ and $\quad$ $B''^\mu(x') = \sum_\nu \dfrac{\partial x'^\mu(x)}{\partial x^\nu} B^\nu(x)$

where

(5) $\qquad\qquad x'^\mu = x'^\mu(x) \quad$ and $\quad x^\nu = x^\nu(x')$

are the general coordinate transformations or GCT. One should notice that such a transformation is local and completely general, except for the fact that it has to be continuously differentiable. That means that the GCT must be continuous and should not present any discontinuity at any order of derivation. The GCT can fluctuate rapidly but all derivatives must necessarily remain continuous. That means, for instance, that discrete coordinate transformations (reflections) are not allowed. As we know, the covariant and contravariant components associated with a vector in an orthogonal Euclidian reference frame are identical. If the Euclidian reference frame is not orthogonal, the contravariant and the covariant components are defined as the projections of the vector normal and parallel to the reference axes respectively. Of course, if the axes of the reference frame are normal, the parallel and the normal projections are identical.

In general, p contravariant and q covariant tensors are defined as

(6) $T'^{\mu'_1\ldots\mu'_p}_{\nu'_1\ldots\nu'_q}(x') = \displaystyle\sum_{\mu_1\ldots\mu_p,\nu_1\ldots\nu_q} \dfrac{\partial x'^{\mu_1}(x)}{\partial x^{\mu_1}} \cdots \dfrac{\partial x'^{\mu_p}(x)}{\partial x^{\mu_p}} \dfrac{\partial x^{\nu_1}(x')}{\partial x''^{\nu_1}} \cdots \dfrac{\partial x^{\nu_q}(x')}{\partial x''^{\nu_q}} T^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q}(x)$

At this point we should make a few remarks about the general coordinate transformations. It can be shown [5-8] that tensorial calculus is valid if and only if the general coordinate transformations are continuously differentiable which means that the transformations are continuous and smooth at any order. Furthermore, the mapping between the original coordinates and the transformed coordinates should be biunivoque which means that to each point corresponds one and only one transformed point, and vice versa. These constraints ensure that, not only the tensorial equations are valid, but that an object cannot be transformed arbitrarily into another object. This is a fundamental requirement for searching and retrieval. Any transformation that satisfies the above mentioned requirements is compatible with our approach.

We shall admit the following results without demonstration [5-8]. The metric is a symmetric tensor $g_{\mu\nu}(x) = g_{\nu\mu}(x)$. If a tensor is identically zero in a coordinate system, it is equal to zero in any other coordinate system. The product of a tensor by a tensor is a tensor and so is the sum. The symmetry and antisymmetry properties of a tensor are conserved under general coordinate transformations and so are the symmetry properties of the corresponding object. In addition, the most important property can be stated as follows: a tensorial equation does not change form under general coordinate transformations. Such a feature is highly desirable if one seeks to define quantities that are coordinate transformations invariant i.e. quantities that can describe an object irrespectively of its state of transformation. Furthermore, the following relations shall be of use

(7) $$\sum_{\rho} \frac{\partial x'^{\mu}(x)}{\partial x^{\rho}} \frac{\partial x^{\rho}(x')}{\partial x'^{\nu}} = \delta_{\nu}^{\mu}$$

(8) $$g_{\mu\nu}(x) = \sum_{\rho} \frac{\partial x'^{\rho}(x)}{\partial x^{\mu}} \frac{\partial x'^{\rho}(x)}{\partial x^{\nu}}$$

(9) $$\sum_{\rho} g^{\mu\rho}(x) g_{\nu\rho}(x) = \delta_{\nu}^{\mu}$$

The metric has an additional important property; it can transform covariant indices into contravariant indices and vice versa as illustrated by the following equations

(10) $$A_{\mu\nu}(x) = \sum_{\alpha\beta} g_{\mu\alpha}(x) g_{\nu\beta}(x) A^{\alpha\beta}(x)$$

The derivative of a tensor is not a tensor. Indeed, if one calculates the derivative of a covariant vector one obtains

(11) $$\frac{\partial A'_{\mu}(x')}{\partial x'^{\nu}} = \sum_{\rho\sigma} \frac{\partial x^{\rho}(x')}{\partial x'^{\mu}} \frac{\partial x^{\sigma}(x')}{\partial x'^{\nu}} \frac{\partial A_{\rho}(x')}{\partial x'^{\sigma}} + \sum_{\rho} A_{\rho}(x') \frac{\partial^2 x^{\rho}(x')}{\partial x'^{\mu} \partial x'^{\nu}}$$

The first term of the right member has the correct form as defined by equations (4) i.e. it is transformed as a tensor, but the second term is incompatible with the definition of a tensor (the transformation involves the second order derivative of $x^{\rho}(x')$). Nevertheless, one can define a covariant derivative, which is form invariant under general coordinate transformations

(12) $$\nabla_\nu A_\mu(x) \equiv \frac{\partial A_\mu(x)}{\partial x^\nu} - \sum_\sigma \Gamma^\sigma_{\mu\nu}(x) A_\sigma(x)$$

where $\Gamma^\sigma_{\mu\nu}$ is called the affine connection and is related to the metric by the following relation

(13) $$\Gamma^\sigma_{\mu\nu}(x) = \sum_\rho \tfrac{1}{2} g^{\sigma\rho}(x) \left(\frac{\partial g_{\mu\rho}(x)}{\partial x^\nu} + \frac{\partial g_{\rho\nu}(x)}{\partial x^\mu} - \frac{\partial g_{\mu\nu}(x)}{\partial x^\rho} \right)$$

It should be noticed that the affine connection is not a tensor. Irrespectively on the mathematics involved, the covariant derivative is a simple concept. The derivative in Euclidian space is related to the concept of slope or in other words the difference in between two points *at two distinct positions*. Such an operation is not problematical in standard Euclidian geometry since the space is flat, homogeneous and isotropic. In our case, the space is not flat but curved and one cannot compare two points at two different locations because they live so to say in two different spaces. What can be done though, is to transport one of the vector "parallel" to itself to the other location and then to compare them on *same* location. This is what is expressed by equation (12) and the affine connection is responsible for such a parallel transportation.

Riemannian spaces are not conservative: if a vector is moved along a close path, the resulting vector does not coincide in general with the original vector. That means again that it is not possible to compare two tensors at two distinct positions. One can easily convince oneself of that. For example, it suffices to move a vector from the North Pole to the equator along a meridian, then once more along the equator and finally back to the North Pole along a meridian: the initial and the final vector point in different direction although their norms are the same. Such a phenomenon happens because the Earth surface is a 2-D curved surface: a sphere.

It becomes interesting then to characterise such behaviour by analysing the variation to which a vector is submitted if it is transported along an infinitesimal loop. For an infinitesimal closed path *C*, one can demonstrate that the variation is proportional to the curvature tensor

(14) $$\Delta A_\mu(x) = -\oint_C \sum_{\nu\sigma} \Gamma^\sigma_{\mu\nu}(x) A_\sigma(x) dx^\nu = \tfrac{1}{2} \sum_{\nu\sigma\tau} R^\nu_{\mu\sigma\tau}(x) A_\nu(x) \, dx^\sigma \wedge dx^\tau$$

where $\wedge$ represents the external product and where the Riemannian curvature tensor $R^{\nu}_{\mu\sigma\tau}(x)$ is defined as

$$(15) \quad R^{\nu}_{\mu\sigma\tau}(x) \equiv \frac{\partial \Gamma^{\nu}_{\mu\tau}(x)}{\partial x^{\sigma}} - \frac{\partial \Gamma^{\nu}_{\mu\sigma}(x)}{\partial x^{\tau}} + \sum_{\rho} \Gamma^{\nu}_{\rho\sigma}(x)\Gamma^{\rho}_{\mu\tau}(x) - \Gamma^{\nu}_{\rho\tau}(x)\Gamma^{\rho}_{\mu\sigma}(x)$$

From the inner product between the metric and the Riemannian curvature tensor, one can define the Ricci tensor $R_{\nu\sigma}(x)$ and the Ricci scalar $R(x)$ which are respectively given by

$$(16) \qquad\qquad R_{\nu\sigma}(x) = \sum_{\mu\tau\rho} g^{\mu\rho}(x) g_{\mu\tau}(x) R^{\tau}_{\nu\rho\sigma}(x)$$

$$(17) \qquad\qquad R(x) = \sum_{\nu\sigma} g^{\nu\sigma}(x) R_{\nu\sigma}(x)$$

The Ricci tensor is symmetric. The Ricci tensor and scalar satisfy many identities among which are the Bianchi identities which is more or less a conservation identity:

$$(18) \qquad\qquad \sum_{\alpha} \nabla_{\alpha}\left(R^{\sigma\alpha}(x) - \tfrac{1}{2} g^{\sigma\alpha} R(x)\right) = 0$$

The Riemann tensor and the Ricci tensor and scalar characterise the curvature of the space at a given point. The Riemann and the Ricci tensor are not invariant under general coordinate transformations: they transform as tensors. However, the Ricci scalar is an invariant: its value is the same irrespectively of the general coordinate transformation applied. Such a feature is common to all scalars in a Riemannian space. At this stage, it is important to realise that the curvature is not bonded to a particular coordinate system but to the physical point itself. For instance, even if the Ricci scalar is an invariant, the coordinates of the point to which it is attached change under a GCT. In section 5, we will see how we can represent invariant quantities independently of their coordinates.

There is a relation in between the Ricci scalar and the standard intrinsic Gaussian curvatures. One can demonstrate that in 2-D, the Ricci scalar and the intrinsic Gaussian curvatures are related by the relation

$$(19) \qquad\qquad {}^{(2)}R(x) \propto \kappa_{1}(x)\,\kappa_{2}(x)$$

where $\kappa_{1}(x)$ and $\kappa_{2}(x)$ are the intrinsic Gaussian curvatures. Such a relation does not exist in 3-D or in higher dimensions. From that point

of view, the Ricci scalar can be considered to a generalisation of the intrinsic curvatures to three dimensions and more.

4   Association of a Curved Space with an Object

In this section, a Riemannian curved space is associated with an object. A set of equations that are form invariant under general coordinate transformations are derived.   In order to construct those equations, a tensor is associated with the object.   Such an association can be realized, for instance, through the density of a tomographic image. This tensor acts as a source term in a field equation from which the geometry of the associated curved space is calculated.

The formulation of form invariant equations under general coordinate transformations is a complex task.   It would be much easier if one could associate and define an invariant scalar functional from which the equations could be derived.   Such an approach has been developed: the scalar functional is the Lagrangian and the equations are derived from a variational principle or least action principle.   For our purpose, the Lagrangian is a scalar functional related to the "energy of the system". The energy could be related to the density of a tomographic image, the 3-D shape deformation (like a deformable mesh) or some topological characteristics (like the number of holes in a certain neighbourhood). The reader who is not familiar with Lagrangians and variational principles is referred to [5, 8] for more details.   Once the Lagrangian is defined; one can derive the corresponding equations by finding the extremum for the action associated with the Lagrangian.   We formulate the Lagrangian in such a way that it incorporates all our requirements about the curved space we want to associate with the object as well as all our knowledge (from an indexing and retrieval point of view) about the object itself.

In a Riemannian space, the action S [5-7] is defined as

$$(20) \qquad S \equiv \int d^N x \sqrt{-\mathbf{det}(g_{\mu\nu}(x))} \; \mathcal{L}\left[g_{\mu\nu}(x)\right]$$

where $\mathcal{L}$ is the Lagrangian (strictly speaking the Lagrangian density) and $\mathbf{det}(g_{\mu\nu}(x))$ the determinant of the metric (the metric is a matrix). One should notice that the action is also a scalar and consequently

invariant under a GCT. The extra factor $\sqrt{-\mathbf{det}(g_{\mu\nu}(x))}$ is related to the Jacobian of the transformation and ensures that the result of the integration does not depend on a particular choice of the coordinate system; in other words, the infinitesimal volume element (or hypervolume) does not depend on the reference frame employed.

The principle of least action [5-8] states that if the action is extremal, the Lagrangian necessarily satisfies the Euler-Lagrange equations, which can be written in our specific case as

(21)
$$\sum_{\rho} \frac{\partial}{\partial x_{\rho}} \frac{\delta\left(\sqrt{-\mathbf{det}(g_{\mu\nu}(x))}\ \mathcal{L}\left[g_{\mu\nu}(x)\right]\right)}{\delta\left(\dfrac{\partial g_{\mu\nu}(x)}{\partial x_{\rho}}\right)} - \ldots$$

$$\frac{\delta\left(\sqrt{-\mathbf{det}(g_{\mu\nu}(x))}\mathcal{L}\left[g_{\mu\nu}(x)\right]\right)}{\delta g_{\mu\nu}(x)} = 0$$

where $\dfrac{\delta h\left[f(x)\right]}{\delta f(x)}$ stands for the functional derivative (the derivative is calculated with respect to a function).

We are now in position to set our hypothesis and derive the corresponding equations. Let us assume that our Lagrangian can be split into two Lagrangians. The first Lagrangian $\widehat{\mathcal{L}}\left[g_{\mu\nu}(x)\right]$ depends solely on the metric and characterises the Riemannian space per se (the space we want to associate with our object) while the second Lagrangian $\breve{\mathcal{L}}\left[g_{\mu\nu}(x),\Phi(x)\right]$ depends on the metric and some other tensor Φ that characterise the object under consideration (for instance the density in a volumetric image). It is very important to understand this point, the space associated with an object is not static but dynamic and its configuration depends on the energy content of the associated object. An analogy, although imperfect, is the association of a magnetic field to a current. As the current is the source of the magnetic field, the object is the source of the associated curved Riemannian space.

With these hypotheses in mind, the action can be written as

(22)
$$S = \int d^N x \sqrt{-\mathbf{det}(g_{\mu\nu}(x))} \left(\widehat{\mathcal{L}}\left[g_{\mu\nu}(x) \right] + \breve{\mathcal{L}}\left[g_{\mu\nu}(x), \Phi(x) \right] \right)$$

Let us write the Lagrangian for the associated curved space. We have seen earlier that a curved space can be characterised by a set of curvatures: the simplest one being the Ricci scalar. Consequently, one of the simplest Lagrangian that can be constructed from the Riemannian curvatures is the one constructed from the Ricci scalar:

(23)
$$\widehat{\mathcal{L}}\left[g_{\mu\nu}(x) \right] = \kappa^{-1} R(x) = \kappa^{-1} \sum_{\mu\nu} g_{\mu\nu}(x) R^{\mu\nu}(x)$$

where κ is an arbitrary constant. Of course this is not the only possibility. One could take, for instance, the tensorial product of a covariant and contravariant Ricci tensor but that would lead to unnecessarily complicated equations. For our purpose, we will be satisfied with the simplest form possible. If one substitutes equation (23) into equation (22) and optimized the action with equation (21) one obtains

(24)
$$\delta S = 0 \quad \Rightarrow \quad R_{\mu\nu}(x) - \tfrac{1}{2} g_{\mu\nu}(x) R(x) - \kappa \, \breve{T}_{\mu\nu}(x) = 0$$

where $\breve{T}_{\mu\nu}$ the source tensor is associated with the object and is defined as

(25)
$$\breve{T}_{\mu\nu}(x) \equiv \frac{\delta \left(\sqrt{-\mathbf{det}(g_{\mu\nu}(x))} \, \breve{\mathcal{L}}\left[g_{\mu\nu}(x), \Phi(x) \right] \right)}{\delta g_{\mu\nu}(x)}$$

Because of equations (18) and (24), the source tensor satisfies the Bianchi identities and is symmetric. In other words, only those source tensors for which the covariant divergence is zero are acceptable. Consequently, when defining the source tensor, one has to be very careful in order to verify that the covariant divergence of equation (25) is effectively zero.

Next, one can demonstrate that the source tensor is related to the density, the momentum and the flux of momentum. For instance, for a static volumetric image one can define the source tensor as

(26)
$$\breve{T}_{00}(x) = \rho(x)$$

and zero otherwise i.e. that the tensor is simply related to the density. In the general case, the source tensor is more complicated. More details can be found in [5-8] but the general approach is well known. One defines a Lagrangian that characterises the energy content of the object under consideration. Such a characterisation might be either

physical (for instance real physical density), topological or formal. Then the source tensor is calculated from equation (25).

Finally, if one substitutes the value of the source tensor in equations (24) one obtains:

(27) $$R_{\mu\nu}(x) - \tfrac{1}{2} g_{\mu\nu}(x)\, R(x) = \kappa \breve{T}_{\mu\nu}(x)$$

which is a set of ten (because of the symmetry properties of the tensors) form invariant non-linear equations describing the relations in between the source tensor associated with the object and the curvatures of the corresponding Riemannian space. These are the relations we were looking for; we have associated a curved space to the object.

In addition, it can be demonstrated [5-8] that the mapping in between the source tensor (i.e. the object) and the Riemannian space is unique and consequently not ambiguous. This result is valid as long as the general coordinate transformations and the source tensor are continuously differentiable. That does not mean that the transformations cannot vary rapidly, it only means that there should be no discontinuities (in the mathematical senses) in the transformations.

The solution of equations (27) is a highly non trivial task. Nevertheless, a numerical solution can be obtained by foliating the space; see for instance [9].

5   Definition of Invariant Representations from a Statistical Representation and from an Abstract Graph

Up to this point, we have associated a Riemann space to an object and we have characterised the curvature of this space by calculating the Ricci tensor and scalar distributions. Now, in order to obtain an invariant description, we must construct some invariant quantities from the Ricci curvatures. If one applies a coordinate transformation to the Ricci scalar, one obtains with the help of equations (7) and (17):

(28) $$R' = \sum_{\mu\nu} g'^{\mu\nu} R'_{\mu\nu} = R$$

Equation (28) shows that the Ricci scalar is invariant under arbitrary coordinate transformations and as a result we define our first ensemble of invariant quantities $\Re_1(x)$ as the ensemble

(29)
$$\left\{ \Re_1(x) | \Re_1(x) \equiv R^2(x) \right\}$$

If one computes the tensorial product of a covariant and a contravariant Ricci tensor one obtains:

(30)
$$\sum_{\mu\nu} R'_{\mu\nu} R'^{\mu\nu} = \sum_{\mu\nu\rho\sigma} \left(\frac{\partial x^\mu}{\partial x'^\rho} \frac{\partial x^\nu}{\partial x'^\sigma} \right) R_{\mu\nu} \left(\frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} \right) R^{\mu\nu} = \sum_{\mu\nu} R_{\mu\nu} R^{\mu\nu}$$

which is again invariant under arbitrary coordinate transformations. Consequently, we define our second ensemble of invariant quantities $\Re_2(x)$ as

(31)
$$\left\{ \Re_2(x) | \Re_2(x) \equiv \sum_{\mu\nu} R_{\mu\nu}(x) R^{\mu\nu}(x) \right\}$$

As a result, an invariant statistical representation of the object can be constructed. The ensembles defined by equations (29) and (31) are described by two histograms. The first histogram characterises the distribution of the Ricci scalars while the second histogram characterises the distribution of the inner products of the Ricci tensors. More precisely, the histograms are defined as

(32)
$$h_k(i) \equiv \sum_{\left\{ x \,\middle|\, \left[\left(i\Delta_k - \frac{\Delta_k}{2} \right) \le \Re_k(x) < \left(i\Delta_k + \frac{\Delta_k}{2} \right) \right] \right\}} \Re_k(x)$$

where Δ_k is the width of each bin for histogram k. In other words, the histograms provide a statistical distribution for the invariants: they do not depend on the location of the invariants on the object but only on their statistical distribution. Such a distribution is invariant under a CGT and characterises the object.

For retrieval purpose, these histograms can be considered as feature vectors and compared with standard techniques such as those described in [1-2]. For instance, comparison can be performed with a metric (distance), a correlation technique, a neural network or with a Bayesian approach. Besides, whatever the method employed, it is important that a certain degree of cross-correlation (bins with different indexes) be present in the comparison algorithm because the invariants, as defined by equations (29) and (32), may possibly present a certain bin index tolerance due to noise and inadequate sampling which means that bins could be shifted and the corresponding histograms distorted. For the metric approach, such a requirement can be implemented with a quadratic form.

An abstract graph representation is also possible. For such a graph, each point for which invariants are calculated is mapped to a node. Each node is related to the pair of invariants calculated at the corresponding point and not to the coordinates of the points, which are in any case arbitrary. The only relations that are invariant, irrespectively of the GCT applied to the object, are the adjacency relations in between the points. Such topological relations remain always the same, because the general coordinate transformations are continuously differentiable by hypothesis. The graph is then constructed is such a way that adjacent nodes (i.e. points) are connected by lines or links. The link indicates only a connection in between two nodes; the length of the link has no meaning per se, since the representation has to be invariant under a GCT. Such a graph is invariant under a GCT. The abstract graph obtained can be compared to other graph using standard techniques [1-2].

The histogram representation is much more compact and is adapted to very large databases. The compactness is obtained at the price of losing the adjacency relations. The abstract graph approach preserves those relations, but the size of the graph limits its applicability to small subset of data for which a detailed representation might be needed.

6. Practical Considerations

The proposed method may be applied to a wide class of 3-D objects. Nevertheless, there are some restrictions that should be taken into account; for instance, the objects under consideration should be Riemannian manifolds. In essence, a manifold is a surface (or a volume) that can be defined by a set of overlapping patches. The surface, including in the overlapping regions, should be continuously differentiable. Such a case is approximated, for instance, by the NURBS or non-uniform rational β-splines which are widely utilised in computer graphics. The approximation stems from the fact that, in the NURBS representation, the overlapping regions are differentiable only up to a certain order. In addition to be a manifold, the surface should be Riemannian. That means that the surface should not present any torsion or, in other words, should not be twisted. For instance, if one

cuts a circular band, twists the two extremities and assembles them back together, one obtains a surface with torsion which cannot be described by the present approach. Otherwise, there are no restrictions and the considered surface can present holes, missing polygons or other types of degeneracy.

For the vast majority of cases of interest, equation (27) must be solved numerically. As pointed out in [9], this is a difficult task in the sense that equation (27) is a set of 10 non-linear differential equations. It has been shown [9] that such a set of equations can be numerically unstable if the numerical algorithm is not carefully designed: for instance, some constrains, like the Bianchi identities i.e. equation (18), must be enforced throughout the calculation. That means that for any practical application the calculation of the invariant representation must be performed offline. On the other hand, the retrieval operation can be performed in real-time since the later involves only the comparison of histograms or graphs for which many real-time comparison approaches exist [1].

At this point, we would like to provide an illustrative example in order to better understand the proposed approach. Let us assume that we have a 3-D object for which we have calculated the invariants as defined by equations (28) to (32). We would like to understand better the meaning of a GCT and how it generalises the traditional approaches. For instance, most invariant representations for 3-D objects are rotation invariant. That means that a unique invariant description can be obtained independently of the orientation of the object in space. Such invariance is global since the object is rotated as a rigid solid. With our approach, it is possible to generalise this invariance to local rotations. By a local rotation we mean that the associated rotation matrix is a function of the coordinates on the object. Let us consider a GCT as defined by equation (4). Such an equation may be expressed in a matrix form as follow

$$
(33) \quad \begin{bmatrix} A'_0(x') \\ A'_2(x') \\ A'_3(x') \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x_0(x')}{\partial x'_0} & \dfrac{\partial x_1(x')}{\partial x'_0} & \dfrac{\partial x_2(x')}{\partial x'_0} \\ \dfrac{\partial x_0(x')}{\partial x'_1} & \dfrac{\partial x_1(x')}{\partial x'_1} & \dfrac{\partial x_2(x')}{\partial x'_1} \\ \dfrac{\partial x_0(x')}{\partial x'_2} & \dfrac{\partial x_1(x')}{\partial x'_2} & \dfrac{\partial x_2(x')}{\partial x'_2} \end{bmatrix} \begin{bmatrix} A_0(x) \\ A_1(x) \\ A_2(x) \end{bmatrix}
$$

The transformation matrix, which in fact is a matrix functional, is invertible by construction since the matrix elements are continuously differentiable. This transformation is extremely general in the sense that the invariant representation does not depend on the form of this matrix. As a matter of fact, this matrix belongs to the group (in the mathematical sense) GL (3) of invertible matrices. We can consider a subgroup of $GL(3)$: for instance all the matrices for which the inverse is equal to the transpose of the transformation matrix. Such a matrix is the rotation matrix i.e. the group $O(3)$ of orthogonal matrices. Consequently, we have demonstrated that our approach is not only invariant for local rotations but also for much more general transformations. Consequently, our approach is a generalisation of global rotation invariance to local rotation invariance; in other words to local deformations.

7. Experimental Results

In this section, we present experimental results. Our objective is to better understand invariants (29) and (31). Firstly, we prove that they are invariant under a general coordinate transformation by explicitly applying such a transformation. Then, we evaluate invariant (29) for particular symmetries of the source tensor. We do not present any evaluation of invariant (31) because they are too cumbersome. These calculations are performed for both 3-D and 4-D objects, in order to better understand the differences in between the two. All the results that follow have been obtained symbolically with the Wolfram Research Mathematica™ software. All the calculations were performed without any approximation. Consequently, the obtained results are exact. They can be utilised either as analytical expressions or as

formulas in numerical evaluations. The results are presented with the Mathematica™ notation [10].

Firstly, we want to prove that our invariants are indeed invariant under a general coordinate transformation or GCT. For this purpose, we apply a GCT to invariant (29) and (31). The calculation is completely general and is performed for both 3-D and 4-D objects.

First, let us consider the case of three-dimensional objects. If we make explicit the summation, invariant (29) could be written as

$$(34) \qquad (R_{11})\ (R^{11}) + 2\ (R_{21})\ (R^{12}) + (R_{22})\ (R^{22})$$

If we apply a GCT (x' = f1[x, y], y' = f2[x, y]) to this invariant we obtain

$$\Big(\big((R^{22})\ f1^{(0,1)}[x, y]^2 + 2\ (R^{12})\ f1^{(0,1)}[x, y]\ f1^{(1,0)}[x, y] +$$
$$(R^{11})\ f1^{(1,0)}[x, y]^2\big)$$
$$\big((R_{11})\ f2^{(0,1)}[x, y]^2 - 2\ (R_{21})\ f2^{(0,1)}[x, y]\ f2^{(1,0)}[x, y] +$$
$$(R_{22})\ f2^{(1,0)}[x, y]^2\big)\Big) \Big/$$
$$(f2^{(0,1)}[x, y]\ f1^{(1,0)}[x, y] - f1^{(0,1)}[x, y]\ f2^{(1,0)}[x, y])^2 +$$
$$\Big(\big((R_{11})\ f1^{(0,1)}[x, y]^2 - 2\ (R_{21})\ f1^{(0,1)}[x, y]\ f1^{(1,0)}[x, y] +$$
$$(R_{22})\ f1^{(1,0)}[x, y]^2\big)\ \big((R^{22})\ f2^{(0,1)}[x, y]^2 + 2\ (R^{12})$$
$$f2^{(0,1)}[x, y]\ f2^{(1,0)}[x, y] + (R^{11})\ f2^{(1,0)}[x, y]^2\big)\Big) \Big/$$
$$(f2^{(0,1)}[x, y]\ f1^{(1,0)}[x, y] - f1^{(0,1)}[x, y]\ f2^{(1,0)}[x, y])^2 +$$
$$(2\ (f1^{(1,0)}[x, y]\ ((R_{21})\ f2^{(0,1)}[x, y] - (R_{22})\ f2^{(1,0)}[x, y]) +$$
$$f1^{(0,1)}[x, y]$$
$$(-(R_{11})\ f2^{(0,1)}[x, y] + (R_{21})\ f2^{(1,0)}[x, y]))$$
$$(f1^{(1,0)}[x, y]\ ((R^{12})\ f2^{(0,1)}[x, y] + (R^{11})\ f2^{(1,0)}[x, y]) +$$
$$f1^{(0,1)}[x, y]$$
$$((R^{22})\ f2^{(0,1)}[x, y] + (R^{12})\ f2^{(1,0)}[x, y]))) \Big/$$

$$(35) \qquad (f2^{(0,1)}[x, y]\ f1^{(1,0)}[x, y] - f1^{(0,1)}[x, y]\ f2^{(1,0)}[x, y])^2$$

where (1,0) indicates a partial derivative with respect to y and x. A similar notation applies to other derivatives. Expression (35) reduces, after simplification, to expression (34) which proves the invariance of (29). One should notice that we need only two coordinates for a three-dimensional object since the later is a surface in three dimensions that can be parameterised with two and only two parameters.

Let us consider the case of 4-D objects. If we make explicit the summation, invariant (29) can be written as

$$(R_{11})\ (R^{11})\ +\ 2\ (R_{21})\ (R^{12})\ +\ 2\ (R_{31})\ (R^{13})\ +$$
$$(36) \qquad (R_{22})\ (R^{22})\ +\ 2\ (R_{32})\ (R^{23})\ +\ (R_{33})\ (R^{33})$$

If we apply a GCT (x' = f1[x, y, z], y' = f2[x, y, z], z' = f3[x, y, z]) to this invariant, we obtain a lengthy expression (10 pages) which, simplifies to (36) after a tedious calculation. Once more, we need three coordinates because a 4-D object is a volume that can be parameterised with three and only three coordinates.

We now calculate invariant (29) for some particular cases. It is possible to perform an exact calculation for the invariant if some kind of symmetry is assumed for the source tensor and consequently for the metric. We consider both 3-D and 4-D objects.

We first address the case of 3-D objects. In the particular case of a three-dimensional object, invariant (29) can be calculated for a general metric. In that case, only two coordinates are needed since a 3-D object is a surface that can be parameterised with two coordinates. If we perform the calculations we obtain

$$\Big(g11[x,\ y]\ \big(g11^{(0,1)}[x,\ y]\ g22^{(0,1)}[x,\ y]\ -$$
$$2\ g22^{(0,1)}[x,\ y]\ g21^{(1,0)}[x,\ y]\ +\ g22^{(1,0)}[x,\ y]^2\big)\ +$$
$$g21[x,\ y]\ \big(g22^{(0,1)}[x,\ y]\ g11^{(1,0)}[x,\ y]\ +$$
$$2\ g21^{(1,0)}[x,\ y]\ (2\ g21^{(0,1)}[x,\ y]\ -\ g22^{(1,0)}[x,\ y])\ -$$
$$g11^{(0,1)}[x,\ y]\ (2\ g21^{(0,1)}[x,\ y]\ +\ g22^{(1,0)}[x,\ y]))\ +$$
$$2\ g21[x,\ y]^2\ \big(g11^{(0,2)}[x,\ y]\ -\ 2\ g21^{(1,1)}[x,\ y]\ +$$
$$g22^{(2,0)}[x,\ y]\big)\ +\ g22[x,\ y]\ \big(g11^{(0,1)}[x,\ y]^2\ +$$
$$g11^{(1,0)}[x,\ y]\ (-2\ g21^{(0,1)}[x,\ y]\ +\ g22^{(1,0)}[x,\ y])\ -$$
$$2\ g11[x,\ y]\ (g11^{(0,2)}[x,\ y]\ -$$
$$2\ g21^{(1,1)}[x,\ y]\ +\ g22^{(2,0)}[x,\ y]))\big)^2\Big/$$
$$(37) \qquad \Big(4\ (g21[x,\ y]^2\ -\ g11[x,\ y]\ g22[x,\ y])^4\Big)$$

Equation (37) reduces to

$$\Big(g11[x,\ y]\ \big(g11^{(0,1)}[x,\ y]\ g22^{(0,1)}[x,\ y]\ +\ g22^{(1,0)}[x,\ y]^2\big)\ +$$
$$g22[x,\ y]$$
$$\big(g11^{(0,1)}[x,\ y]^2\ +\ g11^{(1,0)}[x,\ y]\ g22^{(1,0)}[x,\ y]\ -$$
$$2\ g11[x,\ y]\ (g11^{(0,2)}[x,\ y]\ +\ g22^{(2,0)}[x,\ y]))\big)\Big)^2\Big/$$
$$(38) \qquad (4\ g11[x,\ y]^4\ g22[x,\ y]^4)$$

for the simpler case of a diagonal metric.

We now consider the case of 4-D objects. Let us assume that the metric is diagonal and that the first two elements are equal i.e. that the metric is of the form diag (g11[x, y, z], g11[x, y, z], g33[x, y, z])

With this assumption, invariant (29) can be written as

$$\left(2\, g33[x,\, y,\, z]^2 \left(g11^{(0,1,0)}[x,\, y,\, z]^2 + g11^{(1,0,0)}[x,\, y,\, z]^2\right) -\right.$$
$$g11[x,\, y,\, z]\, g33[x,\, y,\, z]$$
$$\left(-g11^{(0,0,1)}[x,\, y,\, z]^2 + 2\, g33[x,\, y,\, z]\right.$$
$$\left(g11^{(0,2,0)}[x,\, y,\, z] + g11^{(2,0,0)}[x,\, y,\, z]\right)\Big) +$$
$$g11[x,\, y,\, z]^2 \left(2\, g11^{(0,0,1)}[x,\, y,\, z]\, g33^{(0,0,1)}[x,\, y,\, z] +\right.$$
$$g33^{(0,1,0)}[x,\, y,\, z]^2 + g33^{(1,0,0)}[x,\, y,\, z]^2 -$$
$$2\, g33[x,\, y,\, z]\, \left(2\, g11^{(0,0,2)}[x,\, y,\, z] +\right.$$
$$\left.\left.\left.g33^{(0,2,0)}[x,\, y,\, z] + g33^{(2,0,0)}[x,\, y,\, z]\right)\right)\right)^2 \Big/$$
$$(39) \qquad (4\, g11[x,\, y,\, z]^6\, g33[x,\, y,\, z]^4)$$

We need three coordinates to describe a 4-D object, since the later is a volumetric image. If all the diagonal elements are equal, i.e. if the metric is of the form diag (g11[x, y, z], g11[x, y, z], g11[x, y, z]), one obtains

$$\left(3\, g11^{(0,0,1)}[x,\, y,\, z]^2 - 4\, g11[x,\, y,\, z]\, g11^{(0,0,2)}[x,\, y,\, z] +\right.$$
$$3\, g11^{(0,1,0)}[x,\, y,\, z]^2 - 4\, g11[x,\, y,\, z]$$
$$g11^{(0,2,0)}[x,\, y,\, z] + 3\, g11^{(1,0,0)}[x,\, y,\, z]^2 -$$
$$(40) \qquad \left. 4\, g11[x,\, y,\, z]\, g11^{(2,0,0)}[x,\, y,\, z]\right)^2 \Big/ (4\, g11[x,\, y,\, z]^6)$$

which is of course a much simpler expression. The level of complexity of the expression is not only related to the components of the metric tensor (and consequently the source tensor) but also to the level of symmetry of the later.

Finally, let us assume a traceless metric (i.e. all the diagonal elements are equal to zero) without any other restriction on the other elements. Then, invariant (29) is given by the following complex expression

$$\Big(2\,g31[x,\,y,\,z]\,g32[x,\,y,\,z]$$
$$\big(g32[x,\,y,\,z]\,g21^{(1,0,0)}[x,\,y,\,z]\,(g21^{(0,0,1)}[x,\,y,\,z]\,+$$
$$g31^{(0,1,0)}[x,\,y,\,z]\,-\,g32^{(1,0,0)}[x,\,y,\,z])\,+$$
$$g31[x,\,y,\,z]\,g21^{(0,1,0)}[x,\,y,\,z]\,(g21^{(0,0,1)}[x,\,y,\,$$
$$z]\,-\,g31^{(0,1,0)}[x,\,y,\,z]\,+\,g32^{(1,0,0)}[x,\,y,\,z]))\,+$$
$$2\,g21[x,\,y,\,z]^2\,(g31[x,\,y,\,z]\,g32^{(0,0,1)}[x,\,y,\,z]$$
$$(-g21^{(0,0,1)}[x,\,y,\,z]\,+\,g31^{(0,1,0)}[x,\,y,\,z]\,+$$
$$g32^{(1,0,0)}[x,\,y,\,z])\,+\,g32[x,\,y,\,z]$$
$$(-g21^{(0,0,1)}[x,\,y,\,z]\,g31^{(0,0,1)}[x,\,y,\,z]\,+$$
$$g31^{(0,0,1)}[x,\,y,\,z]$$
$$(g31^{(0,1,0)}[x,\,y,\,z]\,+\,g32^{(1,0,0)}[x,\,y,\,z])\,+$$
$$2\,g31[x,\,y,\,z]\,(g21^{(0,0,2)}[x,\,y,\,z]\,-$$
$$g31^{(0,1,1)}[x,\,y,\,z]\,-\,g32^{(1,0,1)}[x,\,y,\,z])))\,+$$
$$g21[x,\,y,\,z]\,\big(2\,g32[x,\,y,\,z]^2\,g31^{(1,0,0)}[x,\,y,\,z]$$
$$(g21^{(0,0,1)}[x,\,y,\,z]\,+\,g31^{(0,1,0)}[x,\,y,\,z]\,-$$
$$g32^{(1,0,0)}[x,\,y,\,z])\,+\,2\,g31[x,\,y,\,z]^2$$
$$(g21^{(0,0,1)}[x,\,y,\,z]\,g32^{(0,1,0)}[x,\,y,\,z]\,-$$
$$g31^{(0,1,0)}[x,\,y,\,z]\,g32^{(0,1,0)}[x,\,y,\,z]\,-$$
$$2\,g32[x,\,y,\,z]\,g21^{(0,1,1)}[x,\,y,\,z]\,+$$
$$2\,g32[x,\,y,\,z]\,g31^{(0,2,0)}[x,\,y,\,z]\,+$$
$$g32^{(0,1,0)}[x,\,y,\,z]\,g32^{(1,0,0)}[x,\,y,\,z]\,-$$
$$2\,g32[x,\,y,\,z]\,g32^{(1,1,0)}[x,\,y,\,z])\,+$$
$$g31[x,\,y,\,z]\,g32[x,\,y,\,z]$$
$$\big(g21^{(0,0,1)}[x,\,y,\,z]^2\,+\,g31^{(0,1,0)}[x,\,y,\,z]^2\,-$$
$$2\,g31^{(0,1,0)}[x,\,y,\,z]\,g32^{(1,0,0)}[x,\,y,\,z]\,+$$
$$g32^{(1,0,0)}[x,\,y,\,z]^2\,-\,2\,g21^{(0,0,1)}[x,\,y,\,z]$$
$$(g31^{(0,1,0)}[x,\,y,\,z]\,+\,g32^{(1,0,0)}[x,\,y,\,z])\,-$$
$$4\,g32[x,\,y,\,z]\,g21^{(1,0,1)}[x,\,y,\,z]\,-$$
$$4\,g32[x,\,y,\,z]\,g31^{(1,1,0)}[x,\,y,\,z]\,+$$
$$4\,g32[x,\,y,\,z]\,g32^{(2,0,0)}[x,\,y,\,z])\big)\big)\Big)^2\Big/$$
$$\big(16\,g21[x,\,y,\,z]^4\,g31[x,\,y,\,z]^4$$
$$g32[$$
$$x,$$
$$y,$$
$$z]^4\big)$$

(41)

where (2,1,0) is a partial with respect to z, y and x. A similar notation applies to the other derivatives.

Consequently, we have obtained exact expressions for invariant (29) for 3-D objects for a general and a diagonal metric. Moreover, for 4-D objects, we obtained exact expressions for invariant (29), for a diagonal metric for which all the elements are equal, for a diagonal metric for which two elements are equal as well as for a traceless metric.

To conclude this section, we would like to present some numerical experimental results for 3-D objects. In the following, all the objects are described with invariant (29) and with representation (32), i.e. the index or descriptor is a histogram of the square of the Ricci tensor. All our calculations were performed with the Viewpoint Datalab™ Libraries and Collections. This repository consists, in our edition, of 12.150 (twelve thousand) objects of a variety of objects such as cars, planes, human bodies, heads, trees, just to mention a few.

With these examples, we illustrate that our method can retrieve an object that has been submitted to a general coordinate transformation or GCT and that such invariance does not deteriorate the discrimination level. That is one of the reasons why we consider such a large database. In addition, we show that the proposed method can be utilised to retrieve similar objects , i.e. that the method is not limited to identical objects submitted to GCT.
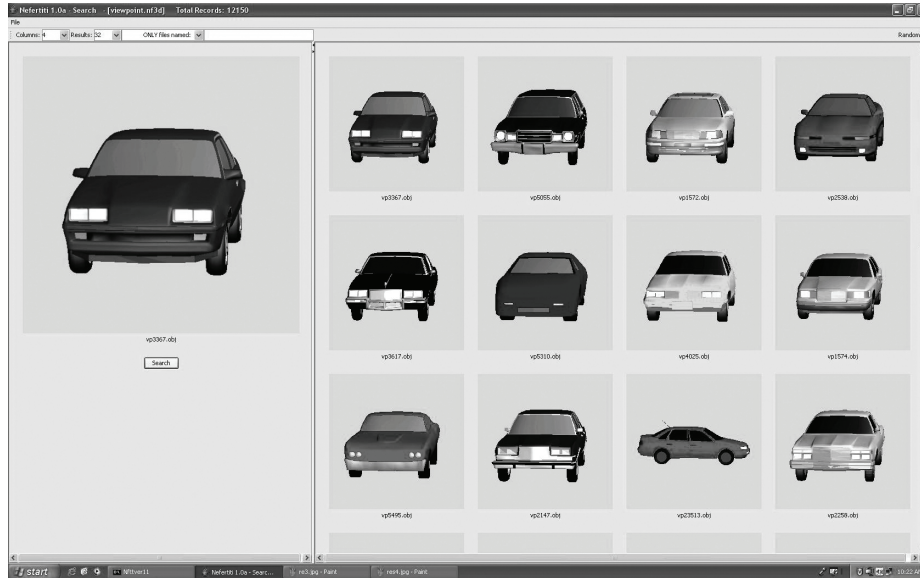
The numerical implementation of the calculation will be the subject of another publication. In essence, we employ a Stochastic or Monte Carlo methods [11] in order to drastically reduce the amount of calculation. The Monte Carlo sampling does not provide an exact result, but an approximation, which, as far as the experimental results are involved, is sufficient for the size (12.150 items) and composition of our database. As a first example, let us consider Figure 2.

Figure 2. Retrieval of an animated 3-D character: the reference object appears on the left side while the outcome of the query appears on the right side. Each result is characterised by a different facial expression i.e. a GCT. In the present query, all the facial expressions of the character were retrieved without any inlayer from a database containing 12.150 objects.
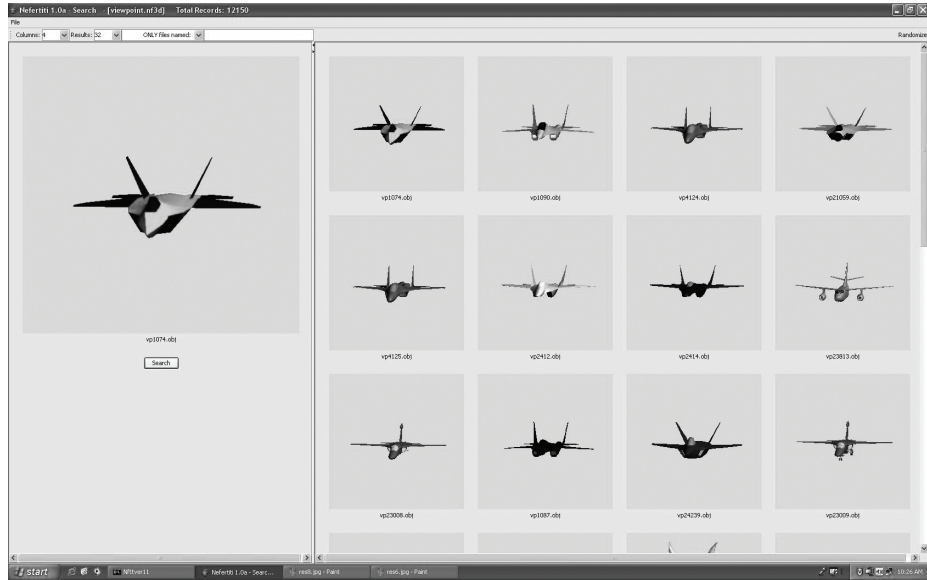
Figure 2 represents a character which was animated with various facial expressions. Such a variation of the facial expression is equivalent to a GCT. We applied our method to this character and retrieved all its facial expressions without any inlayer (precision: 100 %; recall: 100 %). Such a result indicates the efficiency of the method both in terms of invariance under a GCT as well as in terms of discrimination. In our second example, we consider Figure 3, which illustrates a query for a car.

Figure 3. Retrieval of cars. Most of the cars (approximately 90%) were retrieved without inlayers from the 12.150 objects database. Only the first results are displayed.

We managed to retrieve approximately 90% of the cars present in the database without any inlayer (precision: 100%; recall: 90 %). This example shows that the proposed method can be applied, not only to identical objects submitted to a GCT, but also to similar or related objects. Comparable results were obtained for planes and are illustrated in Figure 4. Most of the planes were retrieved without any inlayer, despite the fact that the resolution of the reference model was very low (precision 100 %; recall: 80%).

Figure 4.   Retrieval of planes.   We retrieved most of the planes (approximately 80%) without inlayer despite the fact that the reference model had a very low resolution.   Only the first results are shown.

In our final example, we consider Figure 5 which illustrates a query for an animated body. Here, the woman's arms are in two different positions. Such an animation corresponds to a GCT.

Figure 5. Retrieval of an animated 3-D character. We retrieved all (i.e. 2) the postures associated with the mannequin and most of the human bodies from the 12.150 objects database. Only the first results are shown.

Again, we managed to retrieve both postures without any inlayer. The next retrieved items (up to rank 250) were all human bodies without any inlayer (precision 100 %; recall: 100%). That shows, once more, that the method is invariant under general coordinate transformation and suitable to retrieve similar object while maintaining an adequate discrimination level.

The above-mentioned examples, as many others that are not shown in the present paper, indicate that the proposed method is efficient to retrieve 3-D objects submitted to a GCT as well as similar objects from a large database. The fact that the database is large (12.150 objects) shows, at least from a statistical point of view, that the invariance under GCT does not compromise the level of discrimination of the algorithm.

8. Conclusions

In this paper, we have associated a curved space to an arbitrary object and have described this space with quantities that are invariant under a

general coordinates transformations. From those quantities we have built two representations: one based on the statistical distribution of the invariants and the other based on their topological distribution. Both representations are invariant under GCT. Promising experimental results were provided both analytically and numerically for a database of 12.150 3-D objects.

To the best of our knowledge, there are no approaches that propose such a general and formal framework for GCT invariant representations of object. The next step will be to implement the proposed method, meaning solving exactly equation (27). This will be achieved through a foliation algorithm which will be implemented on a grid computer. In addition, I propose to study various approximations to equation (27) that would be precise enough for indexing and retrieval and that would facilitate and speed up the calculations.

References

1. Iyer, N. et al.: Three-dimensional shape searching: state-of-the-art review and future trends, Computer-Aided Design, Vol. 37 (5) (2005) 509-30.
2. Tangelder, J. W. H. and Veltkamp, R. C.: A Survey of Content Based 3D Shape Retrieval Methods, Proc. IEEE International Conference on Shape Modeling, (2004) 145-56.
3 Zheng, B. et al.: A Survey of 3D Model Retrieval Techniques, Journal of Computer-Aided Design and Computer Graphics, Vol. 16 (5) (2004) 873-881.
4. Vranic, D. V. and Saupe, D.: Description of 3D-Shape using a Complex Function on the Sphere, Proceedings IEEE International Conference on Multimedia and Expo, Vol. 1 (2002) 177-180.
5. Göckeler, M. and Schücker, T.: Differential geometry, gauge theories, and gravity, Cambridge University Press, New York (1989).
6. Rovelli, C.: Quantum Gravity, Cambridge University Press, New York (2004).
7. Kiefer, C.: Quantum Gravity, Oxford University Press, New York (2004).
8. Lovelock David and Rund, Hanno: Tensors, Differential Forms, and Variational Principles, Dover Publications, New York (1989).
9. Bona, C. and Palenzuela-Luque C.: Elements of Numerical Relativity, Springer-Verlag, New York (2005).
10. Wolfram, S.: The Mathematica Book, 5th ed., Wolfram Media, Champaign (2003).
11. Robert, P. R. and Casella, G.: Monte Carlo Statistical Methods, Springer, New York (1999).